Class 61

(8) T^* 1-1 $\Rightarrow T$ dense range (Ex.4.13.3 (b))

Meaning: the existence & uniqueness of solu. of $Tx = y$ & $T^* y^* = x^*$ are related.

Ex. $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on $\mathbb{C}^2$

$$\therefore \begin{cases} ax_1 + bx_2 = y_1 \\ cx_1 + dx_2 = y_2 \end{cases} \text{ has solu } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \forall \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \bar{a}x_1 + \bar{c}x_2 = y_1 \\ \bar{b}x_1 + \bar{d}x_2 = y_2 \end{cases} \text{ solu } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ unique (if exist)}$$

(9) In general, $\overline{\operatorname{ran} T^*} \underset{\neq}{\subset} \ker T^\perp$

Pf.: Let $x^* \in \overline{\operatorname{ran} T^*}$

$\therefore \exists\, y_n^* \in Y^* \ni T^* y_n^* \to x^*$ in $\|\cdot\|$

Check: $\forall x \in \ker T$, $x^*(x) = 0$

$$x^*(x) \uparrow \left(T^* y_n^*\right)(x) = y_n^*(Tx) = 0$$

(10) T^* dense range $\Rightarrow T$ 1-1

Pf.: $\because \overline{\operatorname{ran} T^*} = X^* \subseteq (\ker T)^\perp$ $\Rightarrow (\ker T)^\perp = X^*$ $\Rightarrow (\ker T)^{\perp\perp} = X^{*\perp} = \{0\} \Rightarrow T$ is 1-1 where $(\ker T)^{\perp\perp} = \ker T$ by (5) needs Hahn-Banach Thm

(11) (Banach's closed range thm)

X, Y Banach spaces, $T \in B(X, Y)$.

Then $\operatorname{ran} T$ closed $\Leftrightarrow \operatorname{ran} T^*$ closed.

Moreover, in this case, $\operatorname{ran} T^* = \ker T^\perp$

Lma. X, Y Banach spaces, $T \in B(X, Y)$

If $\operatorname{ran} T$ closed, then $\exists\, K > 0 \ni \forall\, y \in \operatorname{ran} T$, $\exists\, x \in X \ni Tx = y$ & $\|x\| \le K \cdot \|y\|$

Idea: Inverse mapping thm.
Meaning: If T 1-1, then $T: X \to \operatorname{ran}T$ 1-1, onto, conti. & both Banach spaces

$\therefore$ Inverse mapping thm $\Rightarrow T^{-1}: \operatorname{ran}T \to X$ conti.

$\therefore \|x\| = \|T^{-1}y\| \le \|T^{-1}\| \cdot \|y\|$

Pf.: $\because T: X \to \operatorname{ran}T$ onto, linear, bdd & $\operatorname{ran}T$ Banach space

$\tilde{T}(\tilde{x}) = Tx$

$\because \tilde{T}: X/\ker T \to \operatorname{ran}T$ 1-1 onto, bdd

$\|Tx\| = \|T(x+x_1)\| \le \|T\| \cdot \|x+x_1\| \quad \forall x_1 \in \ker T$

$\Rightarrow \|\tilde{T}\tilde{x}\| = \|Tx\| \le \|T\| \cdot \|\tilde{x}\|$

$\Rightarrow \|\tilde{T}\| \le \|T\|$

$\Rightarrow \tilde{T}^{-1}: \operatorname{ran}T \to X/\ker T$ bdd (by Inverse mapping thm)

$\therefore$ For $y \in \operatorname{ran}T$, $\|\tilde{T}^{-1}y\| \le \|\tilde{T}^{-1}\| \cdot \|y\|$

where $y = Tx$ for some x, and $\|\tilde{T}^{-1}y\| = \|\tilde{x}\| = \inf\{\|x+z\| : z \in \ker T\}$

$\therefore \exists\, z \in \ker T \ni \|x+z\| \le \left(\|\tilde{T}^{-1}\|\|y\|\right) + \|y\|$ (May assume $y \ne 0$).

$\therefore \|x+z\| \le \left(\|\tilde{T}^{-1}\| + 1\right) \cdot \|y\|$, where $T(x+z) = Tx = y$.

Pf. of (11). "$\Rightarrow$": Check: $\operatorname{ran}T^* = \ker T^{\perp}$

"$\subseteq$": $\operatorname{ran}T^* \subseteq \overline{\operatorname{ran}T^*} \subseteq \ker T^{\perp}$ by(9)

Check: "$\supseteq$" (need open-mapping thm & Hahn-Banach thm)

Let $x^* \in \ker T^{\perp}$

Define $y^*: \operatorname{ran}T \to F$ by $y^*(Tx) = x^*(x)\ \forall x \in X$

(1) y^* well-defined:

Check: $Tx_1 = Tx_2 \Rightarrow x^*(x_1) = x^*(x_2)$

$T(x_1 - x_2) = 0$

$\Downarrow$

$x_1 - x_2 \in \ker T$

$\Downarrow$

$x^*(x_1 - x_2) = 0$

(2) y^* linear

(3) y^* bdd:

Lma $\Rightarrow \forall y \in \operatorname{ran}T, \exists\, x \in X \ni Tx = y \ \& \|x\| \le K \cdot \|y\|$.

$\therefore \left|y^*(y)\right| = \left|y^*(Tx)\right| = \left|x^*(x)\right| \le \left\|x^*\right\| \cdot \|x\| \le \left\|x^*\right\| \cdot K \cdot \|y\|$

$\Rightarrow \left\|y^*\right\| \le K \cdot \left\|x^*\right\|$

Hahn-Banach Thm $\Rightarrow$ extend y^* to $Y \ni \left\|y^*\right\|$ preserved

$\therefore\ y^* \in Y^*$

$\therefore$ Check: $T^* y^* = x^*$

Check: $\left(T^* y^*\right)(x) = x^*(x)\ \ \forall x \in X$

$\quad\ \ \|$ $\qquad$ //

$\ \ y^*(Tx)$

$\because \operatorname{ran}T^* = (\ker T)^\perp$ is closed.

"$\Leftarrow$": by Ex.4.13.5

Cor. $\operatorname{ran}T$ closed

Then T 1-1 $\Leftrightarrow T^*$ onto. (i.e. uniqueness of $Tx = y \Leftrightarrow$ existence of $T^* y^* = x^*$)

Note: $\|Tx\| \ge \delta \|x\|$ for some $\delta > 0$ & $\forall x \Rightarrow \operatorname{ran}T$ closed.

Pf.: Say, $Tx_n \to y$

Then $\|Tx_n - Tx_m\| \ge \delta \|x_n - x_m\|$

$\Rightarrow \{x_n\}$ Cauchy

Say, $x_n \to x$

$\Rightarrow Tx_n \to Tx$

$\therefore\ y = Tx \in \operatorname{ran}T$

$\Rightarrow \operatorname{ran}T$ closed

Homework:

Sec. 4.13. Ex. 4,5

Sec. 4.14. Conjugate space of L^p :

(F.Riesz, 1918)

Thm. (X, Ω, u) (positive) measure space, $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ ($\Rightarrow 1 < q < \infty$).

Then $L^p(u)^* \cong L^q(u)$

$\begin{pmatrix}\text{isometric} \\ \text{isomorphic}\end{pmatrix}$

Pf.: (1) $\forall g \in L^q(X,u)$, $x_g^*(f) \equiv \int fg du$ defines a bdd linear functional on $L^p(X,u)$ & $\left\|x_g^*\right\| \le \|g\|_q$.

Pf.: x_g^* linear in f. (i.e. $x_g^*(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 x_g^*(f_1) + \alpha_2 x_g^*(f_2)$)

Hölder's ≤ $\Rightarrow$ $\left|\int fg du\right| \le \|f\|_p \cdot \|g\|_q$

$\|$

$\therefore \quad \left|x_g^*(f)\right|$

$\therefore \left\|x_g^*\right\| \le \|g\|_q$

(2) $x_{\alpha g_1 + \beta g_2}^* = \alpha x_{g_1}^* + \beta x_{g_2}^*$

Check: $g \mapsto x_g^*$ is onto.

(cf. J.B.Conway, A course in functional analysis, Appen. B)

(3) $\forall x^* \in L^p(X,u)^*$, $\exists g \in L^q(X,u) \ni x^* = x_g^*$ & $\left\|x^*\right\| = \|g\|_q$

(I) Assume $u(X) < \infty$

$\forall V \in \Omega, \chi_V \in L^p$

Define $\upsilon(V) = x^*(\chi_V) \in F$

Check: (1) $\upsilon(\varnothing) = x^*(\chi_\varnothing) = x^*(0) = 0$

(2) υ countably additive:

Let $\{V_n\}$ disjoint in Ω

Check: $\upsilon\left(\bigcup_n V_n\right) = \sum_n \upsilon(V_n)$

$\|$ $\quad$ $\|$

$x^*\left(\chi_{\bigcup_n V_n}\right) \quad \sum_n x^*\left(\chi_{V_n}\right)$

$\because x^*$ conti.

Check: $\sum_{k=1}^{n} \chi_{V_k} \to \chi_{\bigcup_n V_n}$ in L^p

$\because \int_X \left|\sum_{k=n+1}^{\infty} \chi_{V_k}\right|^p du = \int \sum_{k=n+1}^{\infty} \chi V_k du = \sum_{k=n+1}^{\infty} u(V_k) \to 0$ as $n \to 0$

$\uparrow$

$\because \sum_n u(V_n) = u\left(\bigcup_n V_n\right) < \infty$

$\therefore \upsilon$ (signed) measure

Moreover, $u(V)=0 \Rightarrow \chi_V = 0$ a.e. $\Rightarrow \upsilon(V) = x^*(\chi_V) = x^*(0) = 0.$

$\therefore \upsilon << u$

$\therefore$ Radon-Nikodym Thm $\Rightarrow \exists$ meas. $g \ni \upsilon(V) = \int_V g\,du \ \ \forall V \in \Omega$

$$\begin{array}{ccc} \upsilon(V) & = & \int_V g\,du \\ \| & & \| \\ x^*(\chi_V) & & \int \chi_V g\,du \end{array}$$

In parti., $V = X \Rightarrow g$ integrable on X $\Rightarrow g$ finite a.e.

$\Rightarrow x^*(f) = \int fg\,du \ \forall$ simple f.