Class 62

Check: $g\in L^q(X,u)$ & $\|g\|_q \le \|x^*\|$ & $x^*(f)=\int fg\,du\ \forall f\in L^p(X,u)$

$\forall t>0$, let $E_t=\{x\in X : |g(x)|\le t\}$

(i) Let $f\in L^p$ & $f=0$ on $X\setminus E_t$

Check: $x^*(f)=\int fg\,du$

$\exists f_n$ simple, meas. $\ni f_n\to f$ a.e. & $|f_n|\le|f|$ a.e. $\forall n$

Reason:
$\because \exists g_n \ni 0\le g_n \uparrow f^+$
$h_n \ni 0\le h_n\uparrow f^-$
$\Rightarrow g_n-h_n\to f^+-f^-=f$
$\&|g_n-h_n|\le f^++f^-=|f|$
$|||$
f_n

(1) $\because |f_ng|\le|f|\cdot|g|\le|f|t$ a.e. $\forall n$ ($\because f=0$ on $X\setminus E_t$)
& $\int|f|du=\int|f|\cdot 1du\le\left(\int|f|^p\right)^{\frac{1}{p}}\cdot\left(\int 1^q\right)^{\frac{1}{q}}=\left(\int|f|^p\right)^{\frac{1}{p}}\cdot u(X)^{\frac{1}{q}}<\infty$
(Hölder $\le$)
$\because f_ng\to fg$ a.e. $\Rightarrow \int f_ng\to\int fg$ (DCT)

(2) $\because |f_n-f|^p\le(|f_n|+|f|)^p\le(2|f|)^p=2^p\cdot|f|^p$ integrable
$\therefore |f_n-f|^p\to 0$ a.e.
DCT $\Rightarrow \int|f_n-f|^p\to 0.$
i.e., $f_n\to f$ in $\|\cdot\|_p$
$\Rightarrow x^*(f_n)\to x^*(f)$

$\because x^*(f_n)=\int f_ng$
$\downarrow(2)\qquad\downarrow(1)$
$x^*(f)\qquad \int fg$

$\Rightarrow x^*(f)=\int fg\ \forall f\in L^p \ni f=0$ on $X\setminus E_t$

(ii) Check: $g\in L^q$ & $\|g\|_q\le\|x^*\|$.

Let $A=\{x\in X : g(x)\ne 0\}$.

Let $f_t=\chi_{A\cap E_t}\dfrac{|g|^q}{g}$ for each $t>0$.

Then $f_t = 0$ on $X \setminus E_t$.

Check: $f_t \in L^p$

$\therefore \int |f_t|^p = \int_{A\cap E_t} \frac{|g|^{qp}}{|g|^p} = \int_{A\cap E_t} |g|^{qp-p} = \int_{E_t} |g|^q \le t^q u(E_t) < \infty$

$\because f_t g = \chi_A \cdot \chi_{E_t} |g|^q = \chi_{E_t} |g|^q$

$$\therefore \int_{E_t} |g|^q \overset{\downarrow}{=} \underline{\underline{\int f_t g \overset{(i)\downarrow}{=} x^*(f_t)}} \le \|x^*\| \cdot \|f_t\|_p \overset{\text{by proof above}\downarrow}{=} \|x^*\| \cdot \left(\int_{E_t} |g|^q\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\int_{E_t} |g|^q\right)^{1-\frac{1}{p}} \le \|x^*\|$$

$$\left(\int_{E_t} |g|^q\right)^{1-\frac{1}{p}} = \left(\int_{E_t} |g|^q\right)^{\frac{1}{q}}$$

Let $t \to \infty \Rightarrow 0 \le \chi_{E_t} |g|^q \uparrow |g|^q \Rightarrow \int_{E_t} |g|^q \uparrow \int |g|^q$ ($\uparrow$ MCT)

$\therefore \|g\|_q \le \|x^*\|$ & $g \in L^q$

(iii) Check: $x^*(f) = \int fgdu \ \forall f \in L^p$.

$\because x^*(f) = \int fgdu$ for simple f

$\therefore x^*(f) = \int fgdu \ \forall f \in L^p$

($\uparrow$ as in (i), $\left(\because |f_n g| \le |fg| \in L^1(u)\right)$)