Class 63

(II) X arbitraty

Let $E \subseteq X \ni u(E) < \infty$.

Consider measure space $(E,\ \Omega_E,\ u_E)$

where $\Omega_E = \{\Delta \in \Omega:\ \Delta \subseteq E\}$

$(u_E)(\Delta) = u(\Delta)$ for $\Delta \in \Omega_E$

$\therefore$ Consider $\underset{\cong}{L^p(u_E)} \subseteq L^p(X,u)$

$\{f \in L^p(X,u): f = 0 \text{ on } X \setminus E\}$.

Let $x^* \in L^p(X,u)^*$

Consider $x_E^* : L^p(u_E) \to F :$ restriction of x^* to $L^p(u_E)$.

Then x_E^* linear & $\|x_E^*\| \le \|x^*\|$

$\therefore x_E^* \in L^p(u_E)^*$

$\therefore$ From (I), $\exists g_E \in L^q(u_E) \ni x_E^*(f) = \int_E f g_E du\ \forall f \in L^p(u_E) \Rightarrow \|g_E\|_q = \|x_E^*\| \le \|x^*\|$.

Let $\varepsilon = \{E \subseteq X : u(E) < \infty\}$

Let $D, E \in \varepsilon$.

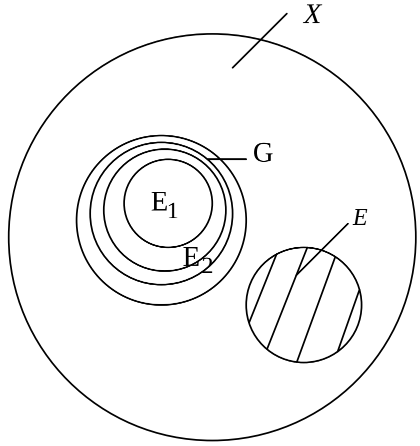
Then $L^p(u_{D\cap E}) = L^p(u_D) \cap L^p(u_E)$

$x_{D\cap E}^* = x_D^* \big| L^p(u_{D\cap E}) = x_E^* \big| L^p(u_{D\cap E})$

$\Rightarrow g_{D\cap E} = \underline{\underline{g_D = g_E}}$ a.e. $[u]$ $\underline{\underline{\text{on } D\cap E}}$ (by uniqueness of g on $D\cap E$).

$\therefore$ Define $g(x) = \begin{cases} g_E(x) & \text{if } x \in E \text{ for some } E \in \varepsilon. \\ 0 & \text{if } x \notin \underset{E\in\varepsilon}{\cup} E \end{cases}$

In parti., $g_E = x_E g$.



(1) Check: g measurable

Motivation: ε may be uncountable $\Rightarrow$ change to countable

Let $\sigma = \sup\left\{\|g_E\|_g : E \in \varepsilon\right\} \le \|x^*\|$.

$\Rightarrow \exists\{E_n\} \subseteq \varepsilon \ni \|g_{E_n}\|_q \to \sigma$.

$\because D \subseteq E \in \varepsilon \Rightarrow \|g_D\|_p = \|x_D^*\| \le \|x_E^*\| = \|g_E\|_q$.

May assume $E_n \uparrow$.

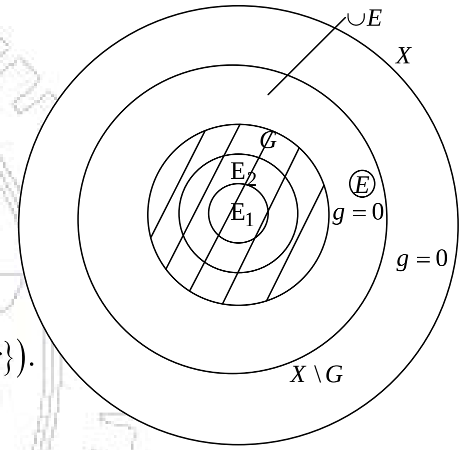
Let $G = \bigcup_n E_n$.

Let $E \in \varepsilon \ni E \cap G = \varnothing$

Then $\|g_{E\cup E_n}\|_q^q = \int |g_{E\cup E_n}|^q = \int |g_E|^q + \int |g_{E_n}|^q = \|g_E\|_q^q + \|g_{E_n}\|_q^q \to \|g_E\|_q^q + \sigma^q \le \sigma^q$

(where $\|g_{E\cup E_n}\|_q^q \le \sigma^q$)

$\because |g_{E\cup E_n}|^q = |g_E|^q + |g_{E_n}|^q$, since $|g_{E\cup E_n}|^q = \chi_{E\cup E_n}|g|^q = \chi_E|g|^q + \chi_{E_n}|g|^q$



$\Rightarrow \|g_E\|_q^q = 0$

$\Rightarrow g = 0$ on E

$\Rightarrow g = 0$ on $X \setminus G = \left[\cup\{E : E \in \varepsilon, E \cap G = \varnothing\}\right] \cup \left(X \setminus \cup\{E : E \in \varepsilon\}\right)$.

(2) $\therefore g_{E_n} = \chi_{E_n} g \to \chi_G g = g$

g_{E_n} meas. $\Rightarrow g$ meas.

$\because \left(\int |g_{E_n}|^q\right)^{\frac{1}{q}} \uparrow \left(\int |g|^q\right)^{\frac{1}{q}}$ (MCT), where $\left(\int |g_{E_n}|^q\right)^{\frac{1}{q}} = \|g_{E_n}\|_q$ and $\left(\int |g|^q\right)^{\frac{1}{q}} = \|g\|_q$

But $\|g_{E_n}\|_q \uparrow \sigma$

$\Rightarrow \|g\|_q = \sigma \le \|x^*\|$

$\Rightarrow g \in L^q$.

(3) Check: $x^*(f) = \int fg \quad \forall f \in L^p$

Let $f \in L^p$

$\because \{x \in X : f(x) \neq 0\}$ is σ- finite (by p.47, Ex.2.6.2)

$\qquad \|$

$\therefore \quad \bigcup_n D_n$, where $D_n \in \varepsilon$ & $D_n \uparrow$

$\therefore \chi_{D_n} f \to \chi_{\bigcup_n D_n} f = f$ in L^p (DCT)

$\Rightarrow x^*\left(\chi_{D_n} f\right) \to x^*(f)$

$\qquad \|$

$x^*_{D_n}\left(\chi_{D_n} f\right)$

$\qquad \|$

$\int_{D_n} fg_{D_n}$

$\qquad \|$

$\int_{D_n} fg \to \int fg$ (DCT)

$\Rightarrow x^*(f) = \int fg \ \forall f \in L^p$

(4) Check: $\|g\|_q = \|x^*\|$

$\therefore \|x^*\| \leq \|g\|_q$ in general for $x^*(f) = \int fg$

$(2) \Rightarrow \|g\|_q \leq \|x^*\|$

Cor. (X, u) measure space, $1 < p < \infty$

Then $L^p(X, u)$ reflexive.

Pf.: (cf. p.180)

Note 1. Let $X = \{1, 2, 3, \ldots\}$ (Ex.4.14.4)

$\Omega = 2^X$

$u =$ counting measure.

Then $L^p(X, u) = l^p$

(1) $\left(l^p\right)^* \cong l^q$ if $1 < p < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$

$\quad \cup \qquad \cup$

$x^* \leftrightarrow (\eta_1, \eta_2, \ldots)$

$\ni x^*(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} \eta_i x_i$

(2) l^p reflexive.

Note 2. (Ex.4.14.5)

$X=\{1,2,...,n\}$

$\Omega=2^X$

$u=$ counting measure

Then $L^p(X,u)=\mathbb{R}^n$

(1) $\left(\mathbb{R}^n,\|\cdot\|_p\right)^*\cong\left(\mathbb{R}^n,\|\cdot\|_q\right)$ if $1<p<\infty$

(2) $\left(\mathbb{R}^n,\|\cdot\|_p\right)$ reflexive $\forall\ 1<p<\infty$

Thm. (X,u) σ-finite measure space.

Then $L^1(X,u)^*\cong L^\infty(X,u)$

$$\begin{array}{ccc} L^1(X,u)^* & \cong & L^\infty(X,u) \\ \Psi & & \Psi \\ x^* & \leftrightarrow & g \end{array}$$

$x^*(f)=\int fg\,du\ \ \forall f\in L^1(X,u)$

(as in the proof of preceding thm.)

Note: Not true if X not σ-finite.

Ex. May even let $X=\{1\},u(X)=\infty,u(\varnothing)=0$

$\Omega=\{\varnothing,X\}$

$L^p(u)=\{0\}\ \ \forall 1\le p<\infty$

$L^\infty(u)=\{a:a\in\mathbb{R}\}=\mathbb{R}$

$\therefore L^p(u)^*=\{0\}\ \ \forall 1\le p<\infty$

$\therefore\{0\}=L^p(u)^*\cong L^q(u)=\{0\}$ holds for $1<p<\infty$

But $L^1(u)^*=\{0\}\ne\mathbb{R}=L^\infty(u)$

Pf.: (I) $u(X)<\infty$:

Check: $g\in L^\infty$ & $\|g\|_\infty\le\|x^*\|$, i.e., Check: $u\left(\left\{x\in X:|g(x)|>\|x^*\|\right\}\right)=0$

(ii) For $\varepsilon>0$, let $A=\left\{x\in X:|g(x)|>\|x^*\|+\varepsilon\right\}$

Check: $u(A)=0\left(\Rightarrow u\left\{x\in X:|g(x)|>\|x^*\|\right\}\right)\le\sum_n u\left\{x\in X:|g(x)|>\|x^*\|+\frac{1}{n}\right\})=0)$.

$$\begin{cases} \text{Let } f=x_{E_t\cap A}\dfrac{\bar{g}}{|g|}, \text{ where } E_t=\left\{x\in X:|g(x)|\le t\right\} \text{ for } t>0. \\ \text{Check: } f\in L^1 \\ \because \|f\|_1=\int|f|=\int_{E_t\cap A}\dfrac{|\bar{g}|}{|g|}=u(E_t\cap A)<\infty \end{cases}$$

$\because \int fg = \int_{E_t \cap A} \frac{\overline{g}}{|g|} \cdot g = \int_{E_t \cap A} |g| \geq \left(\|x^*\| + \varepsilon\right) u\left(E_t \cap A\right)$

$\|$

$x^*(f)$

$\wedge\backslash$

$\|x^*\| \cdot \|f\|_1$

$\|$

$\|x^*\| \cdot u\left(E_t \cap A\right)$

Let $t \to \infty$

$\therefore \|x^*\| \cdot u(A) \geq \left(\|x^*\| + \varepsilon\right) \cdot u(A)$

$\Rightarrow u(A) = 0$

Then follow as before.

(II) X σ-finite:

Note: ess. sup. $\left|g_{E \cup E_n}\right| \neq$ ess. sup. $\left|g_E\right| +$ ess. sup. $\left|g_{E_n}\right| \cdot$ for arbitrary X

Let $X = \bigcup_n E_n$, where $E_n \in \varepsilon$ & $E_n \uparrow$

As before, define $g(x) = \begin{cases} g_{E_n}(x) & \text{if } x \in E_n \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$ & $\|g_{E_n}\|_\infty \leq \|x^*\|$

(1) $\because g_{E_n} = \chi_{E_n} g \uparrow \chi_X g = g$

$\therefore g_{E_n}$ meas. $\Rightarrow g$ meas.

& $\|g\|_\infty = \sup_n \|g_{E_n}\|_\infty \leq \|x^*\|$

$\left(\because X = \bigcup_n E_n\right)$

(2), (3), (4) as before.

Note 1. $X = \{1, ..., n\}, u =$ counting measure $\Rightarrow L^1(X, u)$ reflexive (Need proof)

Note 2. $L^1(X, u)$ not reflexive for $X = \mathbb{R}^n, u =$ Lebesgue measure.

Pf: $\because L^1(X, u)$ sep. (cf. Ex.3.2.2)

$L^\infty(X, u)$ not sep. $\Rightarrow L^\infty(X, u)^*$ not sep.

$\Rightarrow L^1(X, u)$ not reflexive

Note 3. $\left(l^1\right)^* \cong l^\infty$

$x^* \leftrightarrow (\eta_1, \eta_2, ...)$

$x^*(x_1, x_2, ...) = \sum_{i=1}^{\infty} \eta_i x_i$

Note 4. l^1 not reflexive