## Class 64

Thm. $C[0,1]^* \cong NBV \equiv BV_0$ (Normalized & bdd variation) (Riesz, 1909)

$x^* \leftrightarrow g_0$ (isometric isomorphism)

$x^*(f) = \int_0^1 f\,dg_0 \ \forall f \in C[0,1]$

Motivation:

Def. $BV = \{g$ on $[0,1]$ of bdd variation$\}$ (cf. p.54) & $V(g)$ = total variation of g ≡

$$\sup\left\{ \sum_{i=1}^{n} \left| g(x_i) - g(x_{i-1}) \right| : \underset{\underset{x_0}{\|}}{0} \le x_1 < \ldots < x_n = 1 \right\}$$

Then $g = g_1 - g_2$, where $g_1, g_2 \uparrow$ on $[0,1]$ (cf. p.54, Ex.2.8.3)

$\because u_{g_i}((a,b]) = g_i(b) - g_i(a),\ i = 1,2$

$\therefore$ extended to Lebesgue-Stieljes measure $u_{g_i}$, $i = 1,2$

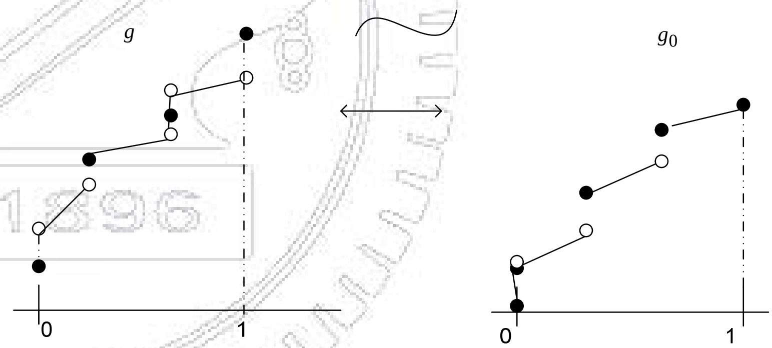
$\therefore \int f\,dg_i, i = 1,2,$ defined

$\therefore \int f\,dg$, defined $= \int f\,dg_1 - \int f\,dg_2$

Def. "~" in BV: $g \sim h$ if $\int_0^1 f\,dg = \int_0^1 f\,dh \ \forall f \in C[0,1]$, equiv,

$\exists$ constant $c \ni g(x) = h(x) + c$ (Ex.4.14.3) for all $x$ except when $g$ or $h$ is disconti. at $x$

Then "~" equivalence relation.




Let $BV_0 = \{[g] : g \in BV\}$

Let $\|[g]\| = \inf_{h \in [g]} V(h)$

Then $(BV_0, \|\cdot\|)$ normed space

Consider the representative $g_0 \in [g] \ni$

(1) $g_0(0) = 0$;

(2) $g_0$ right conti. on $[0,1)$;

(3) $g_0$ left conti. at $t = 1$.

Def. $NBV = \{g_0\}$

Then $NBV$ normed space under $\|g_0\| = V(g_0)$

Note: $g$ of bdd variation

$\Rightarrow g(x+), g(x-)$ exist $\forall x$

& $g$ has only countable jump. disconti.

Thm. $X$ locally compact, top. space (Ex. $X=[0,1]$, $\{1,2,...,n\}$, $\{1,2,...\}$, R)

$C_0(X)=\{f:X\to \mathbb{R} \text{ conti. } \& \ \forall\varepsilon>0,\ \{x:|f(x)|\ge\varepsilon\} \text{ compact}\}$

Then $C_0(X)^*\cong M(X)\equiv\{\text{regular Borel signed measures on } X\}$

Def. $u\ge 0$ is regular Borel if

(1) $\forall K\subseteq X$ compact, $u(K)<\infty$,

(2) $\forall E$ Borel, $u(E)=\sup\{u(K):K\subseteq E \text{ compact}\}$,

(3) $\forall E$ Borel, $u(E)=\inf\{u(U):U\supseteq E \text{ open}\}$.

Def. $u$ signed Borel is regular if

$|u|(E)=\sup\left\{\sum_{i=1}^{n}|u(E_i)|:\{E_i\}_{i=1}^{n} \text{ Borel decomp. of } E\right\}$ (E Borel) is regular

$\|f\|_\infty=\sup\{|f(x)|:x\in X\}$ for $f\in C_0(X)$

$\|u\|=|u|(X)$

$x^*\leftrightarrow u$

$x^*(f)=\int_X f\,du \ \forall f\in C_0(X)$

Cor. 1. $X=\{1,2,...,n\}$

Then $(\mathbb{R}^n,\|\cdot\|_\infty)^*\cong(\mathbb{R}^n,\|\cdot\|_1)$

Cor. 2. $c_0^*\cong l^1$

Let $X=\{1,2,3,...\}$ locally compact top. space.

$C_0(X)=\{(x_n):x_n\to 0\}\equiv c_0$

$M(X)=l^1$

$x^*\in c_0^*\leftrightarrow(y_n)\ni x^*((x_n))=\sum_n x_n.y_n$

Cor. 3. $X=\mathbb{R}$

Then $C_0(\mathbb{R})=\left\{f:\mathbb{R}\to\mathbb{R} \text{ conti., } \lim_{x\to\pm\infty}f(x)=0\right\}$

$\& \ C_0(\mathbb{R})^*\cong M(\mathbb{R})$

Ref. J. B. Conway, A course in functional analysis, 2nd ed., p.383

Thm.1. $\forall x^* \in C[0,1]^*, \exists\, g \in BV \ni x^*(f) = \int_0^1 f\,dg \ \ \forall f \in C[0,1]$ & $\|x^*\| = V(g)$.

Pf.: $\left[\begin{array}{l}\text{Motivation:} \\ \text{Then } x^*\left(\chi_{[0,x]}\right) = \int_0^x dg = g(x) - g(0) \Rightarrow g \\ \text{Difficulty: } \chi_{[0,x]} \notin C[0,1]\end{array}\right]$

$\therefore$ Hahn- Banach Thm $\Rightarrow x^*$ extended to $\Phi \in L^\infty(0,1)^*$ & $\|\Phi\| = \|x^*\|$

Define $g(x) = \Phi\left(\chi_{[0,x]}\right) \ \forall x \in [0,1]$

Check: (1) $g \in BV$ & $V(g) \le \|x^*\|$.

Consider $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$

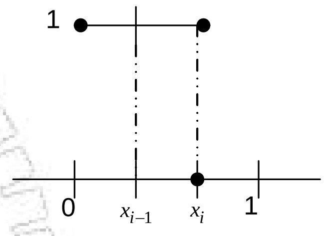
$$\therefore \sum_1^n \left|g(x_i) - g(x_{i-1})\right| = \sum_i \varepsilon_i \cdot \left(g(x_i) - g(x_{i-1})\right), \text{ where } |\varepsilon_i| = 1 \ \forall i$$

$$= \sum_i \varepsilon_i \cdot \left(\Phi\left(X_{[0,x_i]}\right) - \Phi\left(X_{[0,x_{i-1}]}\right)\right)$$

$$= \Phi\left(\sum_i \varepsilon_i \cdot X_{[x_{i-1},x_i]}\right)$$

$$\le \|\Phi\| \cdot \left\|\sum_i \varepsilon_i \cdot X_{[x_{i-1},x_i]}\right\|_\infty$$

$$= \|x^*\| \cdot 1$$




(2) $x^*(f) = \int_0^1 f\,dg . \forall f \in C[0,1]$

Let $f \in C[0,1]$.

Let $f_n(t) = \sum_{k=1}^n f\left(\frac{k}{n}\right)\left(\chi_{\left[0,\frac{k}{n}\right]} - \chi_{\left[0,\frac{k-1}{n}\right]}\right)$

$\because f_n \to f$ in $\|\cdot\|_\infty$ ($\because f$ unif. conti. on $[0,1]$)

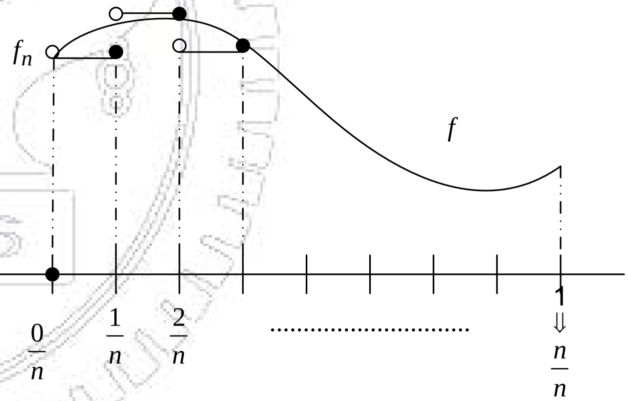
$\Rightarrow \Phi(f_n) \to \Phi(f)$

$\quad\ \ \| \qquad\qquad \|$

$\sum_k f\left(\frac{k}{n}\right)\left(\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right]\right) \quad x^*(f)$

$\downarrow$

$\int_0^1 f\,dg$ (Riemann-Stieltjes integral)




$\Rightarrow x^*(f) = \int_0^1 f\,dg$

(3) $\|x^*\| \le V(g)$:

$$\because \left|x^*(f)\right| = \left|\int_0^1 f\,dg\right| \le \|f\|_\infty \cdot \left|\int_0^1 dg\right| = \|f\| \cdot \left|g(1) - g(0)\right| \le \|f\|_\infty \cdot V(g) = \forall f \in C[0,1]$$

$\Rightarrow \|x^*\| \le V(g)$.

Thm. 2. $C[0,1]^* \cong BV_0$

isometric isom.

$x^* \leftrightarrow [g]$

$x^*(f) = \int_0^1 f\,dg \quad \forall f \in C[0,1]$.

Pf.: $\forall [g]$, define $x^*_{[g]}(f) = \int_0^1 f\,dg \quad \forall f \in C[0,1]$.

Then, (1) well-defined, & $\left\|x^*_{[g]}\right\| \le V(g) \quad \forall g \in [g]$ (Thm 1(3))

$\Rightarrow$ (2) $\left\|x^*_{[g]}\right\| \le \|[g]\|$

(3) $[g] \to x^*_{[g]}$ linear

(4) onto

Thm.1. $\Rightarrow \forall x^*, \exists g \ni x^*(f) = \int_0^1 f\,dg$ & $\left\|x^*\right\| = V(g)$

Consider $[g]$.

(5) isometric:

Then $\left\|x^*\right\| = V(g) \ge \|[g]\|$. Conclusion: $BV_0 \cong C[0,1]^*$

Let $g_0 \in [g]$ be a normalization of $[g]$

Lma.

If $g \in BV[0,1], \exists\, 1\ g_0$ normalized $\ni g \sim \{g_0\}$. Moreover, $V(g_0) = V(g)$.

Pf.: cf. pp.183~184

Let $NBV = \{g_0\}$

Define $\|g_0\| = V(g_0)$.

Then $(NBV, \|\cdot\|)$ normed space

Lma. $(NBV, \|\cdot\|) \cong (BV_0, \|\cdot\|)$

isometric isomorphism

$g_0 \leftrightarrow [g]$

Pf.: $\because \|g_0\| = V(g_0) \ge \|[g]\|$

Check: $\|g_0\| \le \|[g]\|$

i.e. Check: $V(g_0) \le V(g) \quad \forall g \sim g_0$

(cf. p.184)

(Motivation: Look at graphs of $g$ & $g_0$)

Homework:

Sec. 4.14 Ex.6