Class 65

Chap 4: Duality (of spaces & operators)

Chap 5: Spectral theory for compact operators

Chap 6: Spectral theory for normal operators

Chap. 5 Compact operators

X, Y normed spaces, $T : X \to Y$ linear

Def. T is compact if $\forall\ A \subseteq X$ bdd $\Rightarrow \overline{TA}$ compact.

Note 1. T compact $\Leftrightarrow \underline{\underline{\forall\ \{x_n\} \subseteq X \text{ bdd seq. } \exists\ Tx_{n_i} \text{ conv. (in norm).}}}$

Reason: "$\Rightarrow$" Y normed space

Let $A = \{x_n\}$

$\therefore \overline{TA}$ compact $\Leftrightarrow$ sequentially compact.

Note 2. T compact $\Rightarrow T$ bdd

Pf.: Let $A = \{x \in X : \|x\| \le 1\} \Rightarrow \overline{TA}$ compact $\Rightarrow \overline{TA}$ bdd $\Rightarrow T$ bdd.

Note 3. $T : X \to Y$ has finite rank & bdd

$\Rightarrow T$ compact (Ex.5.1.1)

Pf.: Let $A \subseteq X$ be bdd

$(\because \|Tx\| \le \|T\| \cdot \|x\|)$

$\therefore \overline{TA}$ closed & bdd in finite-dim $\overline{TX} = TX$

$\Rightarrow \overline{TA}$ compact.

Note 4. $T : X \to$ □ linear $\not\Rightarrow T$ bdd.

Ex. Let $\{x_\alpha\}$ be Hamel basis of X, $\dim X = \infty$

Let $Tx_{\alpha_n} = n$ & $Tx_\alpha = 0$ for $\alpha \ne \alpha_n$

Then T linear, but unbdd.

Note 5. $T : X \to Y$ linear, $\dim X < \infty \Rightarrow T$ compact.

Pf.: $\dim X < \infty \Rightarrow T$ bdd & rank $T < \infty$ $\Rightarrow T$ compact.

Thm. $T : X \to Y$ compact

Then $\{x_n\}$ weakly conv. $\Rightarrow \{Tx_n\}$ strongly conv.

Pf.: Assume $x_n \to x_0$ weakly.

& $Tx_n \not\to Tx_0$ strongly

Then $\exists\ Tx_{n_k}, \varepsilon > 0 \ni \left\|Tx_{n_k} - Tx_0\right\| \ge \varepsilon\ \forall k$

$\because x_{n_k} \to x_0$ weakly.

$\because \{x_{n_k}\}$ bdd by unif. bddness principle.

Note 1. $\Rightarrow \exists x'_{n_k} \ni Tx'_{n_k}$ conv., say, to y_0 (strongly)

But $x'_{n_k} \to x_0$ weakly

$\Rightarrow Tx'_{n_k} \to Tx_0$ weakly (Ex. 4.10.11)

(Reason: $\forall\, y^* \in Y^*, y^*T \in X^* \Rightarrow \left(y^*T\right)\left(x'_{n_k}\right) \to \left(y^*T\right)\left(x_0\right)$)

$$\underset{y^*\left(Tx'_{n_k}\right)}{\|} \qquad \underset{y^*\left(Tx_0\right)}{\|}$$

Conclusion: $y_0 = Tx_0$

i.e., $Tx'_{n_k} \to Tx_0$ strongly $\to\leftarrow$ $\left\|Tx_{n_k} - Tx_0\right\| \ge \varepsilon \quad \forall k$

Note: T bdd $: X \to X$

Then (1) $x_n \to x_0$ in $\|\cdot\| \Rightarrow Tx_n \to Tx_0$ in $\|\cdot\|$

(2) $x_n \to x_0$ weakly $\Rightarrow Tx_n \to Tx_0$ weakly (Ex. 4.10.11)

(3) $x_n \to x_0$ in $\|\cdot\| \Rightarrow Tx_n \to Tx_0$ weakly: trivial

(4) $x_n \to x_0$ weakly $\not\Rightarrow Tx_n \to Tx_0$ in $\|\cdot\|$

Ex. $X = l^2$

$T = I$

$x_n = (0,\ldots,0,1,0,\ldots),\ n \ge 1.$

n-th

Then $x_n \to 0$ weakly,

but $Tx_n = x_n \not\to 0$ in $\|\cdot\|$

Thm. X normed space, Y Banach space.

$T_n,: X \to Y$ compact, $T: X \to Y$ & $T_n \to T$ in $\|\cdot\|$

Then T compact.

Pf.: Let $A \subseteq X$ be bdd.

Check: $\overline{TA}$ compact

$\Updownarrow$

Check: $\overline{TA}$ totally bdd (cf. p.109; complete metric space)

i.e., $\forall \varepsilon > 0,\ \exists$ finitely many $B\left(x_i, \varepsilon\right) \ni \bigcup_i B\left(x_i, \varepsilon\right) \supseteq \overline{TA}.$

(Idea: Find T_n close to T & use total bddness of $\overline{T_nA}$)

$\therefore \forall \varepsilon > 0, \exists\ T_n \ni \left\|T_nx - Tx\right\| \le \left\|T_n - T\right\| \cdot \underset{K}{\underset{\wedge\backslash}{\|x\|}} < \varepsilon \quad \forall x \in A$

$\because \overline{T_n A}$ compact

$\Rightarrow \overline{T_n A}$ totally bdd

$\therefore \exists\, B(x_i,\varepsilon) \ni \underset{i}{\cup} B(x_i,\varepsilon) \supseteq \overline{T_n A}$.

Check: $\underset{i}{\cup} B(x_i,\varepsilon) \supseteq \overline{TA}$ (cf. Ex.3.5.2)

Let $y \in \overline{TA}$

$\therefore \exists\, Tx,\ x \in A,\ \ni \|y - Tx\| < \varepsilon$

$\& \|Tx - T_n x\| < \varepsilon$

$\& \|T_n x - x_i\| < \varepsilon$ for some i

$\Rightarrow \|y - x_i\| < 3\varepsilon$

$\therefore \overline{TA}$ totally bdd $\Rightarrow$ compact $\Rightarrow T$ compact.

Thm. X normed space.

$T, S.\ X \to X$ operator.

T compact $\Rightarrow TS$ & ST compact.

Pf.: (1) Let $A \subseteq X$ be bdd

$\because S$ operator $\Rightarrow SA$ bdd

T compact $\Rightarrow \overline{TSA}$ compact

$\therefore TS$ compact.

(2) $\because T$ compact $\Rightarrow \overline{TA}$ compact

$\because S$ conti. $\Rightarrow S\overline{TA}$ compact.

But $\overline{STA} \subseteq \overline{S\overline{TA}}$ & closed

$\Rightarrow \overline{STA}$ compact.

Note: Ex.5.1.2 says, S, T compact $\Rightarrow \alpha S + \beta T$ compact

Let X be a Banach space

Let $K = \{T \in B(X) : T \text{ compact}\}$

Then K is a closed ideal containing all finite-rank operators in $B(X)$

Next thm says "K is self-adjoint."

Thm. X Banach space, Y Banach space.

$T : X \to Y$ compact $\Leftrightarrow T^* : Y^* \to X^*$ compact.

Pf.: "$\Rightarrow$": (true for Y normed spaces).

Let $\{y_n^*\} \subseteq Y^*$ bdd.

Check: $\exists\, y_{n_i}^* \ni T^* y_{n_i}^*$ conv. strongly.

(1) Check: $\overline{TX}$ separable

Let $A_n = \{Tx : \|x\| \le n\}$ $\forall n$

$\because$ $\overline{TA_n}$ compact

$\Rightarrow \overline{TA_n}$ separable (cf. p.109; metric space)

$\Rightarrow TA_n \subseteq \overline{TA_n}$ separable ($\because$ metric space by Ex.3,5,7)

$\therefore$ $TX = \bigcup_n TA_n$ separable

$\Rightarrow \overline{TX}$ separable ($\because$ nbd of pt in $\overline{TX}$ contains a pt in TX $\Rightarrow$
nbd of a pt in TX contains a pt in countable dense set)

Let A be dense seq. in $\overline{TX}$.

(2) $\therefore \forall y \in A$, $\{y_n^*(y)\}$ bdd $\Rightarrow \exists \{y_{n_k}^*\} \ni y_{n_k}^*(y)$ conv.

Note: Alaoglu says $\overline{\{y_n^*\}}$ weak-$*$ compact $\not\Rightarrow \overline{\{y_n^*\}}$ weak-$*$ sequen. compact

Diagonal method $\Rightarrow \exists \{y'^*_n\} \ni y'^*_n(y)$ conv. $\forall y \in A$.