Class 66

$\Rightarrow y'^{*}_{n}(y)$ conv. $\forall y \in \overline{TX}$ (cf. Ex.4.12.1)

Pf.: $\because \left|y_n^*(y)-y_m^*(y)\right| \le \left|y_n^*(y)-y_n^*(a)\right| + \left|y_n^*(a)-y_m^*(a)\right| + \left|y_m^*(a)-y_m^*(y)\right|$

$\le \left\|y_n^*\right\| \cdot \left\|y-a\right\| + \left|y_n^*(a)-y_m^*(a)\right| + \left\|y_m^*\right\| \cdot \left\|a-y\right\|$

(3) Let $x_n^* \equiv T^*\left(y_n^*\right) \in X^*$.

Check: x_n^* conv. strongly.

$\because x_n^*(x) = T^*\left(y'^{*}_{n}\right)(x) \underset{\uparrow}{=} y'^{*}_{n}(Tx)$ conv. $\forall x \in X$

(def. of adjoint)

(Need: X Banach space)

Thm.4.5.2. $\Rightarrow \exists x^* \in X^* \ni x_n^*(x) \to x^*(x) \quad \forall x \in X$ ($\because X^*$ Banach space)

i.e. $x_n^* \to x^*$ weakly

Check: $x_n^* \to x^*$ strongly

(4) Assume otherwise

Then $\exists\ \eta > 0$ & $x_n^* \ \ni \ \left\|x_n^* - x^*\right\| \ge \eta \quad \forall n$

Let $x_n \in X$ be $\ni \left\|x_n\right\| = 1$ & $\left|x_n^*(x_n) - x^*(x_n)\right| \ge \frac{1}{2}\left\|x_n^* - x^*\right\| \ge \frac{1}{2}\eta$

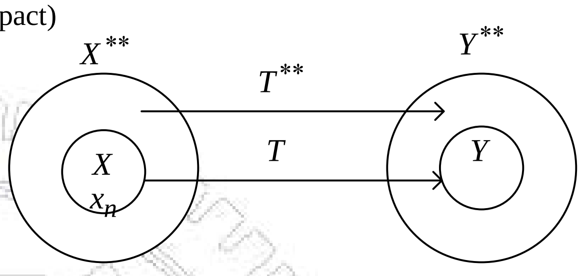
where $x_n^*(x_n) = T^*\left(y_n^*\right)(x_n) = y_n^*(Tx_n)$ and $x^*(x_n) = \lim_m x_m^*(x_n) = \lim_m y_m^*(Tx_n)$

$\begin{cases} \because \left\|x_n\right\| = 1 \text{ \& } T \text{ compact} \\ \Rightarrow \exists\ Tx_{n_j} \to y_0, \text{ say} \\ \forall \varepsilon > 0, \exists\ N \ni n \ge N \Rightarrow \left\|Tx_{n_j} - y_0\right\| < \varepsilon. \end{cases}$

$\begin{cases} \text{On the other hand, } y_0 \in \overline{TX} \Rightarrow y'^{*}_{g}(y_0) \text{ conv.} \\ \therefore n \ge N \Rightarrow \left|y_g^*(y_0) - \lim_m y_m^*(y_0)\right| < \varepsilon \end{cases}$

$$
\begin{aligned}
&\therefore \left| y_g^*\left(Tx_{n_j}\right) - \lim_m y_m^*\left(Tx_{n_j}\right)\right| \\
&\le \left| y_g^*\left(Tx_{n_j}\right) - y_g^*\left(y_0\right)\right| + \left| y_g^*\left(y_0\right) - \lim_m y_m^*\left(y_0\right)\right| + \left| \lim_m y_m^*\left(y_0\right) - \lim_m y_m^*\left(Tx_{n_j}\right)\right| \\
&\le M\cdot\varepsilon + \varepsilon + N\cdot\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \\
&\rightarrow\leftarrow \left| y_g^*\left(Tx_n\right) - \lim_n y_m^*\left(Tx_n\right)\right| \ge \frac{1}{2}\eta.
\end{aligned}
$$

"$\Leftarrow$": (Idea: Restriction of compact operator is compact)



Assume T^* compact.

By "$\Rightarrow$", T^{**} compact

Let $\{x_n\} \subseteq X$ bdd

Check: Tx_{n_i} conv. in Y in norm.

$\because \{\hat{x}_n\} \subseteq X^{**}$ bdd & T^{**} compact

$\Rightarrow T^{**}\hat{x}_{n_i}$ conv. in norm. (Note: $T^{**}: X^{**} \to Y^{**}$).

$\quad\|$

Tx_{n_i} ($\because T^{**}$ extension of T)

$\because Y$ Banach space $\Rightarrow Y$ closed

$\therefore Tx_{n_i}$ conv. (to a limit in Y) in norm.

Homework:

Sec.5.1

Ex.2,5,7,8

Sec.5.2. Fredholm-Riesz-Schauder Theory

X Banach space, T on X compact, $\lambda \ne 0$

Idea: $\lambda I - T$ behaves like operator on finite-dim space.

Lma.1. T compact on X, $\lambda \ne 0$

$\Rightarrow \ker(\lambda I - T)$ finite-dim subspace of X

Pf.: $\because \ker(\lambda I - T)$ closed subspace of X. $\Rightarrow \ker(\lambda I - T)$ Banach space

Let $\{x_n\}$ bdd seq. $\subseteq \ker(\lambda I - T)$

Check: $\exists$ convergent subseq. (cf .p.133, Thm.4.3.3) $\Rightarrow \dim\ \ker(\lambda I - T) < \infty$

$\because T$ compact

$\Rightarrow \exists Tx_{n_j}$ conv. in norm

$\quad\|$

λx_{n_j}

$\Rightarrow x_{n_j}$ conv. in norm

Note 1. T compact, $\lambda \neq 0 \Rightarrow \ker\left(\lambda I - T^*\right)$ finite-dim.

Reason: T compact $\Rightarrow T^*$ compact

Then apply Lma. 1

Note 2. Not true if $\lambda = 0$

Ex. Let $T = 0$ on l^2

Then $\ker T = l^2$ has infinite-dim.

Lma.2. T, λ as above.

Then $\operatorname{ran}(\lambda I - T)$ closed.

Pf.: Let $\{y_n\} \subseteq \operatorname{ran}(\lambda I - T) \ni y_n \to y_0$ in $\|\cdot\|$.

Check: $y_0 \in \operatorname{ran}(\lambda I - T)$

$\because y_n = (\lambda I - T)x_n$ for some $x_n \in X$.

(*) Check: $\exists$ bdd $\{z_n\} \subseteq X$, $\ni y_n = (\lambda I - T)z_n$.