Class 67

Then T compact $\Rightarrow \exists\ Tz_{n_i}$ conv.

$\because\ y_{n_i} = \lambda z_{n_i} - Tz_{n_i}$ conv.

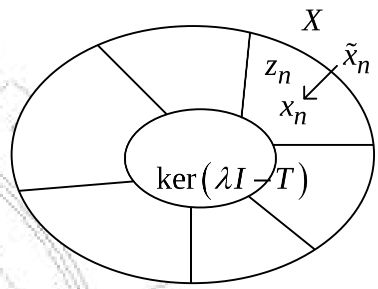
$\Rightarrow \lambda z_{n_i}$ conv.

$\Rightarrow z_{n_i}$ conv., say, to x_0

$$\therefore\ \begin{array}{ccccc} y_{n_i} & = & \lambda z_{n_i} & - & Tz_{n_i} \\ \downarrow & & \downarrow & & \downarrow \\ y_0 & & \lambda x_0 & & Tx_0 \end{array}$$

$\Rightarrow y_0 = (\lambda I - T)x_0 \in \operatorname{ran}(\lambda I - T)$.

Pf. of $(*)$: Consider $X \setminus \ker(\lambda I - T) = \{\tilde{x} : x \in X\}$



Let $\|\tilde{x}_n\| = \inf\{\|x_n - y\| : y \in \ker(\lambda I - T)\}$

Check: $\{\|\tilde{x}_n\|\}$ bdd

Assume otherwise.

$\therefore \exists\ x_{n_j} \ni \|\tilde{x}_{n_j}\| \to \infty$ (unbdd)

Let $\tilde{y}_{n_j} = \dfrac{\tilde{x}_{n_j}}{\|\tilde{x}_{n_j}\|} \Rightarrow \|\tilde{y}_{n_j}\| = \dfrac{1}{\|\tilde{x}_{n_j}\|}\|\tilde{x}_{n_j}\| = 1$

$$\Rightarrow \exists\ v_j \in \ker(\lambda I - T) \ni \Bigg\| \underbrace{y_{n_j} - v_j}_{\substack{\parallel \\ w_j}} \Bigg\| \le 2 \text{ (bdd)}$$

$\begin{cases} \because T \text{ compact} \\ \therefore \exists\ w_{j_k} \ni Tw_{j_k} \text{ conv. in norm} \end{cases}$

$$(**)\therefore (\lambda I - T) w_{j_k} = (\lambda I - T)\left(y_{n_{j_k}} - v_{j_k}\right) = \frac{(\lambda I - T)\left(x_{n_{j_k}}\right)}{\left\|\tilde{x}_{n_{j_k}}\right\|} = \frac{y_{n_{j_k}} \to y_0}{\left\|\tilde{x}_{n_{j_k}}\right\| \to \infty} \to 0$$

$\therefore$ From $(**)$, w_{j_k} conv., say, to w

$\uparrow$

$\because \lambda w_{j_k} - T w_{j_k} \to 0$
$+ \quad T w_{j_k}$ conv.

λw_{j_k} conv. $\Rightarrow w_{j_k}$ conv.

$\therefore \Rightarrow T w_{j_k} \to T w$

$(**) \Rightarrow (\lambda I - T) w_{j_k} \to (\lambda I - T) w$, i.e., $w \in \ker(\lambda I - T)$.

$\downarrow$ 0 $\qquad \Downarrow \|\tilde{w}\| = 0$

But $\left\|\tilde{y}_{n_{j_k}}\right\| = 1$

$\|$

$\left\|\tilde{w}_{j_k}\right\| \to \|\tilde{w}\|$

$\Rightarrow \|\tilde{w}\| = 1 \quad \to\leftarrow$

Say, $\|\tilde{x}_n\| \le C \ \forall n$

$\therefore \exists\, u_n \in \ker(T - \lambda I) \ni \|x_n - u_n\| \le C + 1$

$\qquad |||$ z_n

$\therefore \{z_n\}$ bdd & $y_n = (\lambda I - T) x_n = (\lambda I - T) z_n$, proving $(*)$.

Note 1. Not true if $\lambda = 0$.

Ex. Let $T : l^2 \to l^2 \ni T(x_1, x_2, \ldots) = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\right)$

Then $T_n \to T$ in $\|\cdot\|$, $\left(\because \|T - T_n\| = \frac{1}{n+1} \to 0\right)$

where $T_n(x_1, x_2, \ldots) = \left(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, 0, \ldots\right)$ finite rank $\Rightarrow$ compact

$\Rightarrow T$ compact

But T 1-1, dense range.

If ran$T$ closed, then ran$T = l^2$

But $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in l^2 \setminus \operatorname{ran} T \quad \to\leftarrow$

Note 2: T compact, $\lambda \ne 0 \Rightarrow \operatorname{ran}\left(\lambda I - T^*\right)$ closed

Let $X, T, \lambda \neq 0$ be as before.

Let $N_\lambda^n = \ker(\lambda I - T)^n$ for $n \geq 1$.

Note 1. $N_\lambda^n \subseteq N_\lambda^{n+1}$ $\forall n$

2. $\dim N_\lambda^n < \infty$ $\forall n \geq 1$

Reason: $(\lambda I - T)^n = \underset{\substack{\nparallel \\ 0}}{\lambda^n} I - \underbrace{\binom{n}{1}\lambda^{n-1}T + \binom{n}{2}\lambda^{n-2}T^2 - \ldots + (-1)^n \binom{n}{n} T^n}_{\text{compact}}$

Lma 1 $\Rightarrow \ker(\lambda I - T)^n$ is finite dim.

Lma 3. $\exists\, k \geq 1 \ni N_\lambda^1 \underset{\neq}{\subset} N_\lambda^2 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} N_\lambda^k = N_\lambda^{k+1} = \ldots\ldots$

Note: Not true for $\lambda = 0$

Ex. $T(x_1, x_2, \ldots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\right)$ on l^2

Then $T = T_1 T_2$, where T_1 left shift &

$T_2 =$ multi. by $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

$\because T_2$ compact $\Rightarrow T$ compact.

But $N_0^n = \{(x_1, \ldots, x_n, 0, 0, \ldots)\}$ $\forall\, n \geq 1$

Pf.: (1) Check: $N_\lambda^n = N_\lambda^{n+1} \Rightarrow N_\lambda^{n+1} = N_\lambda^{n+2}$

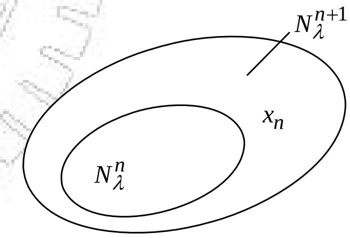
Check: $N_\lambda^{n+2} \subseteq N_\lambda^{n+1}$

Let $x \in N_\lambda^{n+2} = \ker(\lambda I - T)^{n+2}$

$\therefore (\lambda I - T)^{n+2} x = 0$

$\qquad\parallel$

$(\lambda I - T)^{n+1} \underbrace{(\lambda I - T) x}$



$\Rightarrow (\lambda I - T)x \in \ker(\lambda I - T)^{n+1} = \ker(\lambda I - T)^n$

i.e., $(\lambda I - T)^n (\lambda I - T) x = 0$

$\qquad\parallel$

$(\lambda I - T)^{n+1} x$

$\therefore x \in N_\lambda^{n+1}$

(2) Assume $N_\lambda^n \neq N_\lambda^{n+1}\ \forall n.$

Riesz Lma (p.132) $\Rightarrow$ for $\varepsilon = \frac{1}{2}, \exists\ x_n \in N_\lambda^{n+1} \ni \|x_n\| = 1$ & $\|x_n - x\| > 1 - \frac{1}{2} = \frac{1}{2}\ \ \forall x \in N_\lambda^n$

$\because T$ compact

$\Rightarrow \exists\ x_{n_j} \ni Tx_{n_j}$ conv. in norm

But, for $n_j > n_k$

$$Tx_{n_j} - Tx_{n_k} = \left(\lambda I - (\lambda I - T)\right) x_{n_j} - Tx_{n_k}$$

$$= \lambda x_{n_j} - \left(\left(\lambda I - T\right) x_{n_j} + Tx_{n_k}\right) = \lambda\left(x_{n_j} - \frac{1}{\lambda}\left(\left(\lambda I - T\right) x_{n_j} + Tx_{n_k}\right)\right)$$

$$\therefore \left\|Tx_{n_j} - Tx_{n_k}\right\| = |\lambda| \left\| x_{n_j} - \underbrace{\frac{1}{\lambda}\left(\left(\lambda I - T\right) x_{n_j} + Tx_{n_k}\right)}_{\in N_\lambda^{n_j}} \right\| > \frac{1}{2}|\lambda|\ \ \forall n_j > n_k \Rightarrow\ \ \rightarrow\leftarrow$$

Check: $\left(\lambda I - T\right) x_{n_j} + Tx_{n_k} \in N_\lambda^{n_j} = \ker\left(\lambda I - T\right)^{n_j}$ $\ni \left(\lambda I - T\right)^{n_j}\left[\left(\lambda I - T\right) x_{n_j} + Tx_{n_k}\right]$ $= \underbrace{\left(\lambda I - T\right)^{n_j + 1} x_{n_j}}_{=\,0} + \underbrace{T\left(\lambda I - T\right)^{n_j} x_{n_k}}_{=\,0}\ \left(\because x_{n_k} \in N_\lambda^{n_k + 1} \subseteq N_\lambda^{n_j}\right)$

Note: $N_\lambda^1 = \left\{x \in X : \left(T - \lambda I\right) x = 0\right\}$ (eigenspace of λ)

$N_\lambda^k = \bigcup_{n=0}^{\infty} N_\lambda^n = \left\{x \in X : \left(T - \lambda I\right)^n x = 0 \text{ for some } n \geq 1\right\}$ (generalized eigenspace of λ)

$\therefore$ In finite-dim space, $\dim N_\lambda^1 =$ geometric multi. of λ

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$\dim N_\lambda^k =$ algebric multi. of λ

i.e., multi. of λ in characteristic poly. of T

$k =$ size of largest Jordan block of λ.

Ex. $T = \begin{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{bmatrix} & & \\ & \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} & \\ & & [\lambda] \\ & & \end{bmatrix}$

Then $\dim N_\lambda^1 = 3,$

$\dim N_\lambda^k = 6,\ k = 3$