Class 68

Lma. 4. $\lambda I-T$ onto $\Leftrightarrow$ $\lambda I-T$ 1-1.

Pf: "$\Rightarrow$":

Assume $\underbrace{\ker(\lambda I-T)}_{= N_\lambda^1}\neq\{0\}$

Let $x_0\neq 0\in\ker(\lambda I-T)$.

$\because \lambda I-T$ is onto

$\Rightarrow \exists\, x_1 \ni (\lambda I-T)x_1=x_0\neq 0 \ \Rightarrow x_1\notin N_\lambda^1$

$\exists\, x_2 \ni (\lambda I-T)x_2=x_1 \ \because (\lambda I-T)^2 x_2=(\lambda I-T)x_1=x_0\neq 0\Rightarrow x_2\notin N_\lambda^2$

$\vdots$

But $(\lambda I-T)^2 x_1=(\lambda I-T)x_0=0\Rightarrow x_1\in N_\lambda^2$

$(\lambda I-T)^3 x_2=(\lambda I-T)^2 x_1=0\Rightarrow x_2\in N_\lambda^3$

$\vdots$

$\Rightarrow N_\lambda^n\neq N_\lambda^{n+1}\ \forall n \ \rightarrow\leftarrow$

$\therefore \ker(\lambda I-T)=\{0\}$

"$\Leftarrow$":

$\because \operatorname{ran}(\lambda I-T^*)=\ker(\lambda I-T)^\perp=\{0\}^\perp=X^*$

From "$\Rightarrow$", $\ker(\lambda I-T^*)=\{0\}$.

$$\operatorname{ran}(\lambda I-T)=\ker(\lambda I-T^*)^\perp=\{0\}^\perp=X.$$

Note1. "$\Leftarrow$" not true for $\lambda=0$

Ex. $T(x_1,x_2,\ldots)=\left(x_1,\frac{1}{2}x_2,\frac{1}{3}x_3,\ldots\right)$ on l^2

T 1-1, not onto.

Note2. If "$\Rightarrow$" true for $\lambda=0$, then $\operatorname{ran}T=X\Rightarrow T$ 1−1 $\Rightarrow T$ invertible $\Rightarrow I=TT^{-1}$ compact $\Rightarrow \dim X<\infty$.

Lma. 5. $\{x_i^*,\ldots,x_n^*\}$ indep. in X^*

$\Rightarrow \exists\ \{x_1,\ldots x_n\}\subseteq X \ \ni\ x_i^*(x_j)=\delta_{ij}\ \ \forall i,j.$

Pf.: $\because$

Lemma.
$\ker x_j^*\supseteq \bigcap_{i\neq j}\ker x_i^* \Leftrightarrow x_j^*=\sum_{i\neq j}\alpha_i x_i^*$ for some α_i's

(True in any vector space)

$\therefore \ker x_j^*\not\supseteq \bigcap_{i\neq j}\ker x_i^*\ \ \forall j\Leftrightarrow \{x_1^*,\ldots,x_n^*\}$ indep.

Let $y_j \in \left(\bigcap_{i \neq j} \ker x_i^* \right) \setminus \ker x_j^*$

$\therefore x_j^*(y_j) \neq 0$ & $x_i^*(y_j) = 0 \ \forall i \neq j$.

Let $x_j = \dfrac{y_j}{x_j^*(y_j)} \in X$. Then $x_i^*(x_j) = \dfrac{x_i^*(y_j)}{x_j^*(y_j)} = \delta_{ij} \ \forall i, j$

Lemma. $f, f_1, \ldots, f_n$ linear functional on X. Then $\ker f \supseteq \bigcap_{k=1}^{n} \ker f_k$

$\Leftrightarrow f = \sum_{k=1}^{n} \alpha_k f_k$ for some α_k's.

Pf.: "$\Leftarrow$": Trivial.

"$\Rightarrow$": (cf. J.B.Conway, A course in functional analysis, p.377).

May assume $\bigcap_{j \neq k} \ker f_j \neq \bigcap_{j=1}^{n} \ker f_j \ \forall k$

(Reason: Otherwise, $\bigcap_{j \neq k} \ker f_j = \bigcap_{j=1}^{n} \ker f_j \subseteq \ker f$, reduce to $n-1$)

$\therefore \forall k, \exists y_k \in \left(\bigcap_{j \neq k} \ker f_j \right) \setminus \left(\bigcap_{j=1}^{n} \ker f_j \right)$

$= \left(\bigcap_{j \neq k} \ker f_j \right) \setminus \ker f_k$.

$\Rightarrow f_j(y_k) = 0 \ \forall j \neq k$ & $f_k(y_k) \neq 0$

Let $x_k = \dfrac{y_k}{f_k(y_k)}$

$\therefore f_j(x_k) = 0 \ \forall j \neq k$ & $f_k(x_k) = 1$.

Let $\alpha_k = f(x_k) \leftarrow$

Motivation:
If $f(x) = \sum_k \alpha_k f_k(x) \ \forall x$, then $f(x_j) = \sum_k \alpha_k f_k(x_j) = \alpha_j$

Check: $f = \sum_k \alpha_k f_k$.

$\forall x \in X$, let $y = x - \sum_k f_k(x) x_k$.

Then $f_j(y) = f_j(x) - \sum_k f_k(x) f_j(x_k) = 0 \ \forall j$

$\therefore y \in \ker f_j \ \forall j$

$\Rightarrow y \in \ker f$

$\therefore f(y) = 0$

$\qquad \|$

$f(x) - \sum_k f_k(x) f(x_k)$

$\Rightarrow f = \sum_k f_k \cdot \alpha_k$

Lma. 6. $\dim\ker(\lambda I - T) = \dim\ker(\lambda I - T^*) < \infty$.

Note 1. Same as finite-dim operator

2. Not true for $\lambda = 0$

Ex. $T(x_1, x_2, ...) = \left(0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ...\right)$ on l^2.

Then $\dim\ker T = 0$

$\dim\ker T^* = 1$

$$\left(\because T = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \frac{1}{2} & \ddots & \\ & & \ddots & \end{bmatrix} \Rightarrow T^* = \begin{bmatrix} 0 & 1 & & \\ & 0 & \frac{1}{2} & \\ & & \ddots & \ddots \end{bmatrix} \right)$$

Pf.: Let $n = \dim\ker(\lambda I - T)$, $m = \dim\ker(\lambda I - T^*)$.

Assume $n < m$.

Let $\{x_1, ..., x_n\}$ basis in $\ker(\lambda I - T)$

$\{y_1^*, ..., y_m^*\}$ basis in $\ker(\lambda I - T^*)$

Hahn-Banach Thm. $\Rightarrow \exists\, x_1^*, ..., x_n^* \in X^* \ni x_i^*(x_j) = \delta_{ij}\ \forall i, j$

> Reason: $\because x_i \notin \vee\{x_1, ... x_{i-1}, x_{i+1}, ..., x_n\}$
>
> $\Rightarrow \exists\, x_i^* \in X^* \ni x_i^*(x_i) = 1\ \&\ x_i^*(x_j) = 0\ \forall j \neq i$

Lma.5 $\Rightarrow \exists\, y_1, ..., y_m \in X \ni y_i^*(y_j) = \delta_{ij}\ \forall i, j$

(Note: In Hilbert space, let $x_i^* = x_i$ be orthonormal & let $y_i^* = y_i$)

Let $Sx = Tx + \underbrace{\sum_{i=1}^{n} x_i^*(x)\, y_i}_{\text{finite-rank operator}}$ $\quad \forall x \in X$

$\therefore$ S compact.

Check: $\ker(\lambda I - S) = \{0\}$

Let $x \in \ker(\lambda I - S)$

$\therefore Sx = \lambda x$

$\quad \|$

$Tx + \sum_i x_i^*(x)\, y_i$

$\Rightarrow (\lambda I - T)x = \sum_i x_i^*(x)\, y_i$

$$\boxed{\begin{array}{l} \text{Apply } y_j^*: y_j^*\left(\left(\lambda I-T\right)x\right)=\sum_i x_i^*\left(x\right)y_j^*\left(y_i\right)=x_j^*\left(x\right) \quad \forall j \\ \qquad\qquad \| \quad \leftarrow \text{(def. of adjoint)} \\ \qquad \left(\left(\lambda I-T\right)^* y_j^*\right)\left(x\right) \\ \qquad\qquad \| \quad (\because y_j^* \in \ker\left(\lambda I-T^*\right)) \\ \qquad\quad 0 \end{array}} \quad (*)$$

$\Rightarrow \left(\lambda I-T\right)x=0$

i.e., $x\in\ker\left(\lambda I-T\right)$

$\Rightarrow x=\sum_{i=1}^{n}\lambda_i x_i$

Apply x_j^*: $\underset{\underset{0}{\| \text{ (by (*))}}}{x_j^*\left(x\right)}=\sum_i \lambda_i x_j^*\left(x_i\right)=\lambda_j$

$\Rightarrow x=0$

Lma 4 $\Rightarrow \operatorname{ran}\left(\lambda I-S\right)=X$

$\therefore y_{n+1}=\left(\lambda I-S\right)x$ for some $x\in X$

Apply y_{n+1}^*: $y_{n+1}^*\left(\lambda I-S\right)x=y_{n+1}^*\left(y_{n+1}\right)=1$

$\| \text{ (def. of } S)$

$$y_{n+1}^*\left(\left(\lambda I-T\right)x-\sum_i x_i^*\left(x\right)y_i\right)$$

$$\|$$

$$\left(\underset{\underset{0}{\|}}{\left(\lambda I-T^*\right)y_{n+1}^*\left(x\right)}-\sum_i x_i^*\left(x\right)\underset{\underset{0}{\|}}{y_{n+1}^*\left(y_i\right)}\right) \to\leftarrow$$

$\Rightarrow n\ge m$

Applied to $T^*\Rightarrow \underset{\underset{m}{\|}}{\dim\ker\left(\lambda I-T^*\right)}\ge \dim\ker\left(\lambda I-T^{**}\right)\ge \underset{\underset{n}{\|}}{\dim\ker\left(\lambda I-T\right)}$

($\because \lambda I-T^{**}$ extension of $\lambda I-T\Rightarrow \ker\left(\lambda I-T^{**}\right)\supseteq\ker\left(\lambda I-T\right)$).

$\Rightarrow m=n$

Note 1. $\dim\ker\left(\lambda I-T\right)^n=\dim\ker\left(\lambda I-T^*\right)^n<\infty\ \ \forall n\geq 0\ \overset{\text{(Lma. 3)}}{\Rightarrow}\ k_1$ of $T=k_2$ of T^* &

algebraic multi. of λ in $T=$ algebraic multi. of λ in T^*.

geometric multi. of λ in $T=$ geometric multi. of λ in T^*.

Note 2. In Banach spaces: $\lambda^*=\lambda$.

In Hilbert spaces: $\lambda^*=\bar{\lambda}$.

Reason: λI on $X\Rightarrow\left(\lambda I\right)^*$ on X^*.

$$\left(\lambda I\right)^* = \lambda I$$

If identify X^* with X, then $\lambda I\rightarrow\bar{\lambda}I$.

Fredholm alternative:

X Banach space, T compact, $\lambda\neq 0$.

Consider $\left(\lambda I-T\right)x=y$.

Then exactly one of the following alternatives holds:

(1) $\forall y\in X$, $\exists\, 1\ x\in X\ \ni\left(\lambda I-T\right)x=y$.

(2) $\exists\ x\neq 0\in X\ \ni\left(\lambda I-T\right)x=0$.

Moreover, $\left(\lambda I-T\right)x=y$ is solvable in $x\Leftrightarrow y\in\ker\left(\lambda I-T^*\right)^{\perp}$, i.e., $y\ \perp$ finitely many vectors in $\ker\left(\lambda I-T^*\right)$.

↑ (Hence integral equa., dual problem arises naturally)

Pf.: (1) $\Leftrightarrow\lambda I-T$ invertible.

(2) $\Leftrightarrow\lambda I-T$ not 1-1.

By Lma 4., $\lambda I-T$ 1-1 $\Leftrightarrow\lambda I-T$ onto.

Also, $y\in\operatorname{ran}\left(\lambda I-T\right)\Leftrightarrow y\in\ker\left(\lambda I-T^*\right)^{\perp}$

Note 1: True as finite-dim. operators.

Simplest case:

$ax=y$

(1) $\forall y\ \exists\, 1\ x\ni ax=y\left(\Leftrightarrow a\neq 0\right)$

(2) $\exists x\neq 0\ \ni ax=0\left(\Leftrightarrow a=0\right)$

Note 2: Not true for $\lambda=0$:

Ex. $T\left(x_1,x_2,...\right)=\left(x_1,\frac{1}{2}x_2,\frac{1}{3}x_3,...\right)$ on l^2

Then (1) & (2) not true

i.e., T not onto, but 1-1

Note 3. (1) $\Leftrightarrow$ $(1)^*$ $\forall y^* \in X^*, \exists\, 1\, x^* \in X^* \ni \left(\lambda I - T^*\right) x^* = y^*.$

(2) $\Leftrightarrow$ $(2)^*$ $\exists\, x^* \neq 0 \in X^* \ni \left(\lambda I - T^*\right) x^* = 0.$

Moreover, $\left(\lambda I - T^*\right) x^* = y^*$ is solvable in x^* $\Leftrightarrow$ $y^* \in \ker\left(\lambda I - T\right)^{\perp}$.

Pf.: (1) $\Leftrightarrow$ "$\lambda I - T$ invertible $\Leftrightarrow$ $\lambda I - T$ 1-1"

$\Updownarrow$ by Lma 7

$(1)^*$ $\Leftrightarrow$ "$\lambda I - T^*$ invertible $\Leftrightarrow$ $\lambda I - T^*$ 1-1"