Class 69

Sec.5.3. Spectral theory,

X normed space over □.

$T: X \to X$ bdd linear

Def: $\sigma(T) = \{A \in □ : \lambda I - T \text{ not invertible}\}$

(spectrum of T)

$\rho(T) = □ \setminus \sigma(T) = \{A \in □ : \lambda I - T \text{ invertible}\}$

(resolvent set of T).

$R(\lambda, T) = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$

(resolvent of T)

$\lambda \in □$ eigenvalue of T if $\lambda I - T$ not 1-1.

$\lambda \in □$ in continuous spectrum of T if $\lambda I - T$ 1-1, $\overline{\operatorname{ran}(\lambda I - T)} = X$ but $\operatorname{ran}(\lambda I - T) \neq X$.

$\lambda \in □$ in residual spectrum of T if $\lambda I - T$ 1-1, $\overline{\operatorname{ran}(\lambda I - T)} \neq X$

Note: $\sigma(T) = \{\text{eigenvalue}\} \cup$ conti. spectrum $\cup$ residual spectrum, & mutually disjoint.

Thm. X Banach space.

T: operator on X

Then (1) $\rho(T)$ open in □;

(2) $R(u,T) - R(\lambda,T) = (\lambda - u) R(\lambda,T) R(u,T) \ \forall u, \lambda \in \rho(T)$;

(3) $R(\lambda,T)$ analytic for $\lambda \in \rho(T) : \rho(T) \to B(X)$.

Pf.: (1) Let $\lambda_0 \in \rho(T)$.

Check: $B\left(\lambda_0, \dfrac{1}{\left\|(\lambda_0 I - T)^{-1}\right\|}\right) \subseteq \rho(T)$

Check: $|\lambda - \lambda_0| < \dfrac{1}{\left\|(\lambda I - T)^{-1}\right\|} \Rightarrow \lambda I - T$ invertible.

$$\lambda I - T = (\lambda_0 I - T) + (\lambda - \lambda_0) I = \underbrace{(\lambda_0 I - T)}_{\text{invertible}} \underbrace{\left[I + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1} \right]}_{\text{invertible}}$$

(Ex. 4.6.2. on p.144, $\because |\lambda - \lambda_0| \cdot \left\|(\lambda_0 - T)\right\|^{-1} < 1$)

(Need Banach space: $\|A\| < 1 \Rightarrow I + A$ invertible & $(I + A)^{-1} = I - A + A^2 - \ldots$ conv. in $\|□\|$.)

(2) Main idea 通分:

$$\text{LHS} = (uI - T)^{-1} - (\lambda I - T)^{-1} = (uI - T)^{-1}\left[(\lambda I - T) - (uI - T)\right](\lambda I - T)^{-1} \quad (\text{通分})$$

$$= (uI - T)^{-1}(\lambda - u)(\lambda I - T)^{-1} = \text{RHS}.$$

(3) Main idea: 用 (2), reduce to conti.

$$R'(\lambda,T)=\lim_{u\to\lambda}\frac{R(u,T)-R(\lambda,T)}{u-\lambda}=\lim_{u\to\lambda}\frac{(\lambda-u)R(\lambda,T)\cdot R(u,T)}{u-\lambda}$$

(by(2))

$$=-R(\lambda,T)\lim_{u\to\lambda}R(u,T)\underset{\uparrow}{=}-R(\lambda,T)^2$$

$\Rightarrow R(\lambda,T)$ analy. in λ (Reason: As $u\to\lambda, uI-T\to\lambda I-T$ in $\|\cdot\|$

$\Rightarrow (uI-T)^{-1}\to(\lambda I-T)^{-1}$ in $\|\cdot\|$)

Lma. A invertible & $\|B-A\|<\dfrac{1}{\|A^{-1}\|}\Rightarrow B$ invertible & $\|B^{-1}\|\le\dfrac{\|A^{-1}\|}{1-\|B-A\|\|A^{-1}\|}$

Pf: $\|B^{-1}\|\le\|A^{-1}\|\|AB^{-1}\|\le\dfrac{\|A^{-1}\|}{1-\|1-\mathrm{B}A^{-1}\|}\le\dfrac{\|A^{-1}\|}{1-\|B-A\|\|A^{-1}\|}$

Thm. Assume X Banach space over $\mathbb{C}$, $T:X\to X$ bdd operator

(1) $\sigma(T)$ compact (Ex.5.2.6),

(2) $\sigma(T)\ne\varnothing$ (Ex.5.3.1). $\leftarrow$ (Deep: $\dim X<\infty$, by fundamental thm of algebra)

Pf.: (1) $\sigma(T)=\mathbb{C}\setminus\rho(T)$ is closed.

Let $\lambda\in\sigma(T)$

Check: $|\lambda|\le\|T\|$

Assume $|\lambda|>\|T\|$.

Then $\lambda I-T=\lambda\left(I-\frac{T}{\lambda}\right)$ & $\left\|\frac{T}{\lambda}\right\|=\dfrac{\|T\|}{|\lambda|}<1$

$\therefore$ Ex.4.6.2. $\Rightarrow I-\frac{T}{\lambda}$ invertible.

$\Rightarrow \lambda I-T=\lambda\left(I-\frac{T}{\lambda}\right)$ invertible. $\rightarrow\leftarrow$

(2) Assume $\sigma(T)=\varnothing$.

Then $R(\lambda,T)$ analytic on $\mathbb{C}$, i.e. entire func.

Check: $\lim_{|\lambda|\to\infty} R(\lambda,T)=0$.

For $|\lambda|>\|T\|$, $R(\lambda,T)=(\lambda I-T)^{-1}=\frac{1}{\lambda}\left(1-\frac{T}{\lambda}\right)^{-1}$

$=\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^n$ ($\because \left\|\frac{T}{\lambda}\right\|<1$, by Ex. 4.6.2)

$$\therefore \|R(\lambda,T)\|\le \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\frac{\|T\|^n}{|\lambda|^n}=\frac{1}{|\lambda|}\frac{1}{1-\frac{\|T\|}{|\lambda|}}=\frac{1}{|\lambda|-\|T\|}\to 0 \text{ as } |\lambda|\to\infty.$$

$\Rightarrow R(\lambda,T)$ entire & bdd on $\mathbb{C}$.

Liouville's Thm $\Rightarrow R(\lambda,T)$ is constant $\to\leftarrow$

(Same as proving fundamental thm of algebra for finite matrices)

$\Rightarrow \sigma(T)\neq\varnothing$.

Thm. X Banach space, T compact on X, $\dim X=\infty$.

Then one of the following holds:

(1) $\sigma(T)=\{0\}$;

(2) $\sigma(T)=\{0,\lambda_1,\ldots,\lambda_n\}$, where $\lambda_i\neq 0$ eigenvalues;

(3) $\sigma(T)=\{0,\lambda_1,\lambda_2,\ldots\}$, where $\lambda_i\neq 0$ eigenvalues & $\lim_i \lambda_i=0$

[Main idea: (1) indep. of eigenvectors
(2) Riesz Lmma]

Note: If $\dim X<\infty$, $\sigma(T)=\{\lambda_1,\ldots,\lambda_n\}$ can be arbitrary.

Pf.: (i) Check: $0\in\sigma(T)$

i.e., T not invertible.

Assume T invertible

Then $I=T^{-1}T$ compact.

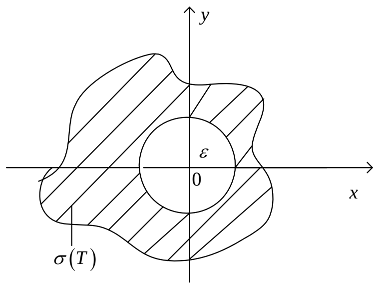
$\Rightarrow \forall$ bdd $Y\subseteq X$, $\overline{I(Y)}=\overline{Y}$ compact

$\therefore$ Thm. 4.3.3. (p.133) $\Rightarrow \dim X<\infty$. $\to\leftarrow$

(ii) $\forall\lambda\in\sigma(T)\setminus\{0\}$, λ eigenvalue of T

Reason: $\lambda I-T$ 1-1 $\Rightarrow$ onto $\Rightarrow$ invertible $\to\leftarrow$

$\therefore \lambda I-T$ not 1-1.



Let $\varepsilon > 0$.

(iii) Check: $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\}$ is finite.

Assume $\lambda_n \in \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\}, n = 1, 2, \ldots$, distinct

$\because \lambda_n \neq 0$

(ii) $\Rightarrow \lambda_n$ eigenvalue of T.

Let $x_n \neq 0$ in $X \ni (\lambda_n I - T)x_n = 0$

Check: $\{x_n\}$ indep. (as in finite-dim space)

Assume otherwise, say, $x_1, \ldots x_{k-1}$ indep. & $x_1, \ldots, x_k$ dependent$(k \geq 1)$

Assume $c_1 x_1 + \ldots + c_k x_k = 0$, with c's not all 0

Apply $T : c_1\lambda_1 x_1 + \ldots + c_k \lambda_k x_k = 0$

$$- \quad \Rightarrow c_1 \frac{\lambda_1}{\lambda_k} x_1 + \ldots + c_{k-1}\frac{\lambda_{k-1}}{\lambda_k} x_{k-1} + c_k x_k = 0$$

$$\underbrace{\left(1 - \frac{\lambda_1}{\lambda_k}\right)}_{\neq 0} c_1 x_1 + \ldots + \underbrace{\left(1 - \frac{\lambda_{k-1}}{\lambda_k}\right)}_{\neq 0} c_{k-1} x_{k-1} = 0$$

$\Rightarrow c_1 = \ldots = c_{k-1} = 0$

$\Rightarrow c_k x_k = 0$

$\Rightarrow c_k = 0 \ \rightarrow\leftarrow$

Let $Y_n = \vee\{x_1, \ldots, x_n\}, n = 1, 2, \ldots$

$$\Rightarrow Y_1 \underset{\neq}{\subset} Y_2 \underset{\neq}{\subset} \ldots\ldots$$

Riesz Lma $\Rightarrow \exists\, y_n \in Y_n \ni \|y_n\| = 1$ & $\|y_n - y\| > \frac{1}{2} \ \forall y \in Y_{n-1}$.

$$\because Y \text{ compact } \& \left\|\frac{y_n}{\lambda_n}\right\| < \frac{1}{\varepsilon} \ \forall n$$

$$\Rightarrow \exists \ \frac{y_{n_j}}{\lambda_{nj}} \ni T\left(\frac{y_{n_j}}{\lambda_{n_j}}\right) \text{ conv.}$$

For $n_k > n_j$,
$$\left\| T\left(\frac{y_{n_k}}{\lambda_{n_k}}\right) - T\left(\frac{y_{n_j}}{\lambda_{n_j}}\right) \right\| = \left\| y_{n_k} - \underbrace{\left(y_{n_k} - T\frac{y_{n_k}}{\lambda_{n_k}} + T\frac{y_{n_j}}{\lambda_{n_j}} \right)}_{\in Y_{n_k - 1}} \right\| > \frac{1}{2} \ \rightarrow\leftarrow$$

Check: $\left(y_{n_k} - T\dfrac{y_{n_k}}{\lambda_{n_k}}\right) + T\dfrac{y_{n_j}}{\lambda_{n_j}} \in Y_{n_k - 1}$

$$\because \boxed{\begin{aligned} & y_{n_k} \in Y_{n_k} \Rightarrow y_{n_k} = \sum_{i=1}^{n_k} \alpha_i x_i \\ & \qquad \Rightarrow Ty_{n_k} = \sum_{i=1}^{n_k} \alpha_i T x_i = \sum_{i=1}^{n_k} \alpha_i \lambda_i x_i \in Y_{n_k} \end{aligned}}$$

$$\Rightarrow y_{n_k} - \frac{1}{\lambda_{n_k}} Ty_{n_k} = \sum_{i=1}^{n_k - 1}\left(1 - \frac{\lambda_i}{\lambda_{n_k}}\right)\alpha_i x_i \in Y_{n_k - 1}$$

$$\& \frac{1}{\lambda_{n_j}} Ty_{n_j} \in Y_{n_j} \subseteq Y_{n_k - 1}$$

(iv) $\sigma(T)$ countable

Reason: $\sigma(T) = \{0\} \cup \bigcup_{n=1}^{\infty} \underbrace{\left(\sigma(T) \cap \left\{\lambda \in \mathbb{C} : |\lambda| > \frac{1}{n}\right\}\right)}_{\text{finite}} \Rightarrow$ countable.

(v) Assume $\{\lambda_i\}$ infinite

Then $\forall \varepsilon > 0,$ except for finitely many λ's, $|\lambda_i| \le \varepsilon$

i.e. $\lim_i \lambda_i = 0$

Homework:

Sec. 5.3 Ex.3,4,10

Ex. for $\sigma(T)$ for compact T.

Ex.1. $T(x_1, x_2, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$ on l^2

Then $\sigma(T) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

Ex.2. $T(x_1, x_2, \ldots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$ on l^2

Then $\sigma(T) = \{0\}$

Pf.: "$\subseteq$": Let $\lambda \in \sigma(T)$ & $\lambda \neq 0$

Then λ eigenvalue of T.

Say, $T(x_1, x_2, ...) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right) = \lambda(x_1, x_2, ...) \neq 0$

$\Rightarrow \lambda x_1 = 0$

$\lambda x_2 = x_1 \qquad \Rightarrow x_1 = x_2 = ... = 0. \ \rightarrow\leftarrow$

$\lambda x_3 = \frac{x_2}{2}$

$\vdots$

$\Rightarrow \sigma(T) = \{0\}$

"$\supseteq$":

$\because T$ not onto

$\Rightarrow T$ not invertible

$\Rightarrow 0 \in \sigma(T)$.

Ex. 3. $T(x_1, x_2, ...) = \left(\frac{x_2}{2}, \frac{x_3}{3}, ..\right)$ on l^2.

Then $\sigma(T) = \{0\}$.

Pf.: "$\subseteq$": Let $\lambda \in \sigma(T)$ & $\lambda \neq 0$

$\therefore \lambda$ eigenvalue of T

$\therefore T(x_1, x_2, ...) = \left(\frac{x_2}{2}, \frac{x_3}{3}, ..\right) = \lambda(x_1, x_2, ...) \neq 0$

$$\Rightarrow \begin{cases} \frac{x_2}{2} = \lambda x_1 & \Rightarrow \ x_2 = 2\lambda x_1 = 2!\lambda x_1 \\ \frac{x_3}{3} = \lambda x_2 & \Rightarrow \ x_3 = 3\lambda x_2 = 3!\lambda^2 x_1 \\ \vdots & \qquad \vdots \\ & \quad x_n = n!\lambda^2 x_1 \end{cases}$$

$\because n!\lambda^n \to \infty$ as $n \to \infty$.

$\therefore (x_n) \in l^2 \Rightarrow x_1 = 0 \ \Rightarrow x_n = 0 \ \ \forall n \ \ \rightarrow\leftarrow$

"$\supseteq$": $\because \ker T = \{(x_1, 0, 0, ...)\} \Rightarrow T$ not 1-1 $\Rightarrow T$ not invertible