Class 7

Thm. u^* metric outer measure on metric space (X,ρ)

$\Rightarrow$ closed (open) sets are measurable.

Note: "$\Leftrightarrow$" true (cf. Ex.1.8.1)

Pf. of Thm.

Let F be closed set

Check: $u^*(A) \geq u^*(A \cap F) + u^*(A \setminus F) \quad \forall A \subseteq X$

Note: $\rho(A \cap F, A \setminus F)$ may not be > 0, replace $A \setminus F$ by smaller E_n

$\because A \setminus F \subseteq F^c$ open

Lma. $\Rightarrow E_n \equiv$

$\left\{ x \in A \setminus F : \rho(x, F) \geq \frac{1}{n} \right\} \ni \lim\limits_{x \to \infty} u^*(E_n) = u^*(A \setminus F).$

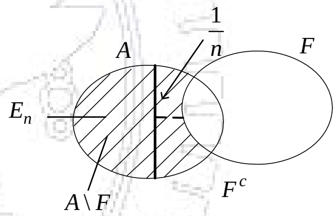
$$\because u^*(A) \geq u^*((A \cap F) \cup E_n) = u^*(A \cap F) + u^*(E_n)$$

$$\uparrow \qquad \downarrow$$

$$(\because \rho(A \cap F, E_n) \geq \rho(E_n, F) \geq \frac{1}{n} > 0) \qquad u^*(A \setminus F) \text{ as } n \to \infty$$

Cor. u^* metric outer measure

$\Rightarrow$ Borel sets are measurable.



Pf. $B = \{\text{Borel sets}\}$

$\boldsymbol{a} = \{\text{measurable sets}\}$

Then $\boldsymbol{a} \supseteq \{\text{closed sets}\}$.

$\Rightarrow \boldsymbol{a} \supseteq \{\text{Borel sets}\}$.

Homework: Ex.1.8.1, 1.8.3, 1.8.4

Sec.1.9. Construction of metric outer measure

(X, ρ) metric space

K sequential convering class

$K_n = \left\{ A \in K : d(A) \leq \frac{1}{n} \right\} \cup \{\phi\}$ for $n \geq 1$.

Assume K_n is a sequential convering class.

Note. In general, false.

Ex. $K = \{[n, n+1] : n \in \mathbb{Z}\} \cup \{\phi\}$ in $\mathbb{R}$.

Ex. $\mathbb{R}$ or $\mathbb{R}^n$

$K = \{\text{open intervals}\} \cup \{\phi\}$

Then K_n sequential convering class $\forall_n$.

$\lambda: K \to [0,\infty]$, $\lambda(\phi)$=0. Then $\lambda \mid K_n \equiv \lambda_n: K_n \to [0,\infty]$.

Let u_n^* outer measure w.r.t. K_n, λ_n, i.e., $u_n^*(A) = \inf\left\{\sum_k \lambda(E_k) : d(E_k) \le \frac{1}{n}\ \forall k,\ A \subseteq \bigcup_k E_k\right\}\ \ \forall A \subseteq X$

Note: 1. $K_{n+1} \subseteq K_n$

2. $u_n^*(A) \le u_{n+1}^*(A) \qquad \forall A \subseteq X$

Define: $u_0^*(A)=\lim u_n^*(A)=\sup u_n^*(A) \quad \forall A \subseteq X$

Thm. u_0^* metric outer measure.

Pf: (1) $u_0^*: \wp(X) \to [0,\infty]$

(2) $u_0^*(\phi) = \lim u_n^*(\phi) = 0.$

(3) $A \subseteq B \Rightarrow u_n^*(A) \le u_n^*(B)\ \forall n$

$$\begin{array}{cc} \downarrow & \downarrow \\ u_0^*(A) & u_0^*(B) \end{array}$$

(4) Countable subadditivity:

Let $E_k \subseteq X \quad \forall k$

$\because u_n^*(\bigcup_k E_k) \le \sum_k u_n^*(E_k) \le \sum_k u_0^*(E_k)$

$\downarrow$

$u_0^*(\bigcup_k E_k)$

$\Rightarrow u_0^*$ outer measure

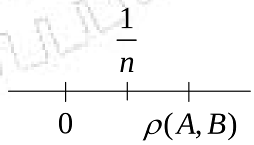
(5) Assume $\rho(A,B) > 0$.

Check: $u_0^*(A)+u_0^*(B) \le u_0^*(A \cup B)$

$\forall \varepsilon > 0,\ \ \forall n,\ \exists \{E_{nk}\} \subseteq K_n \ \ni\ \ A \cup B \subseteq \bigcup_k E_{nk}\ \ \&\ \sum_k \lambda(E_{nk}) \le u_n^*(A \cup B) + \varepsilon.$

$\because d(E_{nk}) \le \frac{1}{n}.$

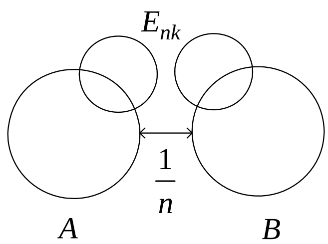
$\because \rho(A,B) > 0 \Rightarrow \rho(A,B) > \frac{1}{n}$ for n large.



$\Rightarrow E_{nk}$ cannot intersect both A, B

Decompose $\{E_{nk}\}$ as $\{E_{nk}'\}$ & $\{E_{nk}''\}$

cover A cover B



$\therefore u_n^*(A) \le \sum_k \lambda(E_{nk}')$

$u_n^*(B) \le \sum_k \lambda(E_{nk}'')$

$\Rightarrow \underline{\underline{u_n^*(A) + u_n^*(B)}} \le \sum_k \lambda(E_{nk}) \le \underline{\underline{u_n^*(A \cup B) + \varepsilon}}$

Let $n \to \infty$ & $\varepsilon \to 0,$ completing the proof.

X metric space

$$\begin{array}{lll} K & \lambda & \to u^* \text{ outer measure} \\ \cup| & \downarrow & \\ K_n & \lambda|K_n \to u_n^* & \uparrow\ u_0^* \text{ metric outer measure} \end{array}$$

Question: $u^* \equiv u_0^*$?

K_n sequential covering class $\forall n \geq 1$

Thm. $\forall A \in K$, $\forall \varepsilon > 0$, $\forall n \geq 1$
$\exists \{E_k\} \subseteq K_n \ni A \subseteq \bigcup_k E_k$ & $\sum_k \lambda(E_k) \leq \lambda(A) + \varepsilon$

Note.
conditions on λ & K
$\Rightarrow u$ metric outer measure

Then $u^* = u_0^*$

Note: condition holds for $\mathbb{R}$ or $\mathbb{R}^n$ (Ex.1.9.3)

$\Rightarrow u^*$ Lebesgue metric outer measure ($\because$ Sec.1.9)

$\Rightarrow$ Borel sets are Lebesgue measurable (Sec.1.8)

Pf: "$\leq$": $K_n \subseteq K$

$$\Rightarrow u^*(A) \leq u_n^*(A) \quad \forall A, \forall n \geq 1$$
$$\downarrow$$
$$u_0^*(A)$$

"$\geq$": $\because \forall A, \forall \varepsilon > 0, \exists \{E_j\} \subseteq K \ni A \subseteq \bigcup_j E_j$ & $\sum_j \lambda(E_j) \leq u^*(A) + \frac{\varepsilon}{2}$

Hypothesis $\Rightarrow \forall E_j$, $\exists \{B_{jk}\} \subseteq K_n \ni E_j \subseteq \bigcup_k B_{jk}$ & $\sum_k \lambda(B_{jk}) \leq \lambda(E_j) + \frac{\varepsilon}{2^{j+1}}$

$\therefore \{B_{jk}\}_{j,k} \subseteq K_n$ & covers A

$$\therefore u_n^*(A) \leq \sum_{j,k} \lambda(B_{jk}) \leq \sum_j \lambda(E_j) + \sum_j \frac{\varepsilon}{2^{j+1}} \leq u^*(A) + \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{= \varepsilon}$$

Let $\varepsilon \to 0$, $n \to \infty$