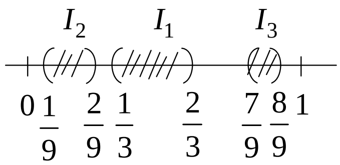
Class 8

Def. $L = \{\text{Lebesgue measurable subets of } \mathbb{R}\}$ ($\rightarrow$ from measure theory)

$B = \{\text{Borel subsets of } \mathbb{R}\}$ ($\rightarrow$ from topology)

$m =$ Lebesgue measurable on $\mathbb{R}$

Relations between L&B:



Def. $C = [0,1] \setminus (I_1 \cup I_2 \cup I_3 \cup \ldots)$ (cf. Ex. 1.9.12)

Cantor set

Properties:

(1) C bdd & closed $\Rightarrow$ compact & Borel

($\because$ intersection of closed sets)

(2) $m(C) = 0$ Pf: $\because$ $m(C) = 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \ldots = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0$

Set Theory: study <u>number of elements</u> of infinite set

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cardinality

G. Cantor (1845-1918)

A, B sets

Def. $\# A = \# B$ if $\exists\ f : A \to B$ 1−1 & onto

$\# A \le \# B$ if $\exists\ f : A \to B$ 1−1

Def. $(\# A) + (\# B) = \#(A \cup B)$ (disjoint union of A & B)

$(\# A) \cdot (\# B) = \#(A \times B)$

$\# A^{\# B} = \#\{ f : B \to A\}$

$\aleph_0 = \# \mathbb{N}$

$\aleph_1 = \# \mathbb{R}$

Thm. 1. $\# \wp(A) = 2^{\# A}$

Thm. 2. $\#A \le \#B$ & $\#B \le \#A \Rightarrow \#A = \#B$ (Schröder-Bernstein)

Thm. 3. $\#A < 2^{\# A}$

Thm. 4. A infinite set $\Leftrightarrow$ A has a subset $C \ni \# A = \# C$

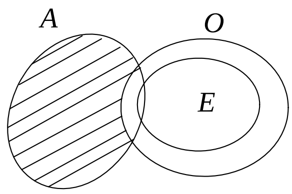
Thm. 5. A infinite set $\Leftrightarrow$ A has a subset $B \ni \# B = \# \mathbb{N}$

Thm. 6. $\aleph_1 = 2^{\aleph_0}$

Note: 1. specific sets in $L \setminus B$ difficult to give. (typical for modern analysis)

2. $m \mid B$ not complete. Reason: $\exists$ subsets of C, not in B. ($\because \#2^C = 2^{\aleph_1} > \aleph_1 = \# B$)

3. m is the completion of $m \mid B$. (see below)



Thm 1. $E \subseteq \mathbb{R}$ (cf. Royden, p. 62)

Then $E \in L \Leftrightarrow \forall \varepsilon > 0, \exists$ open $O \supseteq E \ni m^*(O \setminus E) < \varepsilon$

Pf. "$\Rightarrow$"(Ex. 1.9.7)

"$\Leftarrow$" Check: $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E) \quad \forall A \subseteq \mathbb{R}$

$\because O \in L$

$\because m^*(A) = m^*(A \cap O) + m^*(A \setminus O)$

$\qquad\qquad\qquad \vee\!/ \qquad\qquad \vee\!/$

$m^*(A \cap E) \quad m^*(A \setminus E) - m^*(O \setminus E) \geq m^*(A \setminus E) - \varepsilon$

$\because A \setminus E \subseteq (A \setminus O) \cup (O \setminus E)$

$\because m^*(A \setminus E) \leq m^*(A \setminus O) + m^*(O \setminus E)$

$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E) - \varepsilon$

Let $\varepsilon \to 0$

Thm 2. $E \subseteq \mathbb{R}$

Then $E \in L \Leftrightarrow \forall \varepsilon > 0, \exists$ closed $F \subseteq E \ni m^*(E \setminus F) < \varepsilon$

Pf. "$\Rightarrow$" $E \in L \Rightarrow E^c \in L$

Apply Thm 3 to $E^c \Rightarrow \exists$ open $O \supseteq E^c \ni m^*(O \setminus E^c) < \varepsilon$

$E \supseteq O^c \ni m^*(E \setminus F) = m^*(E \setminus O^c) = m^*(O \setminus E^c) < \varepsilon$

$\quad |||$ $\qquad\qquad\qquad$ $||$ $\qquad$ $||$

$\quad F$ $\qquad\qquad\qquad$ $E \cap O = O \cap (E^c)^c$

closed

"$\Leftarrow$" Check: $E^c \in L$ by reversing above arguments.

Thm 3. $A \subseteq \mathbb{R}$

Then $A \in L \Leftrightarrow A = C \cup N,$ where $C \in B, N \in L$ & $m(N) = 0$

Pf. "$\Leftarrow$" trivial

"$\Rightarrow$" (Ex. 1.9.8)

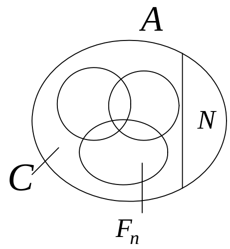
Thm 2. $\Rightarrow \forall n \geq 1, \exists$ closed $F_n \ni F_n \subseteq A$ & $m^*(A \setminus F_n) < \dfrac{1}{n}$

Let $C = \bigcup_{n=1}^{\infty} F_n \in B, C \subseteq A$

Let $N = A \setminus C \in L$

$m(N) = m(A \setminus C) \leq m(A \setminus F_n) < \dfrac{1}{n} \quad \forall n \geq 1$

$\Rightarrow m(N) = 0$



Note.1. m is the completion of $m|B$ & $\overline{B}=L$

2. Thm's 1-3 true for $\mathbb{R}^n$

Littlewood principles (Royden, p. 72, Sec. 3.6)

Principle I: $A \in L \Leftrightarrow A \sim \bigcup_{i=1}^{n} I_i$

Thm 4. $E \subseteq \mathbb{R}$, $m^*(E) < \infty$. (Royden, p. 63)

Then $E \in L \Leftrightarrow \forall \varepsilon > 0$, $\exists$ finite open intervals $\{I_i\}_{i=1}^{n}$ $\ni m^*(E \triangle (\bigcup_{i=1}^{n} I_i)) < \varepsilon$

Pf. "$\Rightarrow$":

Thm 1 $\Rightarrow \forall \varepsilon > 0$, $\exists$ open $O \supseteq E$ $\ni m(O \setminus E) < \varepsilon$

> Note: $O \subseteq \mathbb{R}$, O open $\Leftrightarrow O = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ disjoint open intervals.
>
> Pf. "$\Leftarrow$": trivial
>
> "$\Rightarrow$":
>
> Define $x \sim y$ if $\overline{xy} \subseteq O$ for $x, y \in O$.
>
> Then "$\sim$" equivalence relation.
>
> Each equivalence class is an open interval
>
> $\Rightarrow O = \bigcup_{\alpha} I_\alpha$
>
> $\because$ Correspond each I_α to different rational no. in I_α.
>
> $\Rightarrow \{I_\alpha\}$ countably many
>
> $\therefore O = \bigcup_{n=1}^{\infty} I_n$.

$\because O \setminus \bigcup_{i=1}^{n} \mathrm{I}_i \downarrow \phi$

$\Rightarrow m(O \setminus (\bigcup_{i=1}^{n} \mathrm{I}_i)) < \varepsilon$ for large n

$\Rightarrow m(E \triangle (\bigcup_{i=1}^{n} I_i)) \le m(E \setminus (\bigcup_i I_i)) + m((\bigcup_i I_i) \setminus E)$

$$\begin{array}{cc} \wedge\backslash & \wedge\backslash \\ m(O \setminus (\bigcup_i I_i)) & m(O \setminus E) \\ \wedge & \wedge \\ \varepsilon & \varepsilon \end{array}$$

"$\Leftarrow$" as in Thm 1.

Homework: Ex 1.9.7, 1.9.14, 1.9.15