Class 9

Sec.1.10. Signed measure

$\boldsymbol{a}$ σ-algebra on X

Def: $\mu : \boldsymbol{a} \to (-\infty,\infty]$ or $[-\infty,\infty)$ is a signed measure if

(1) $u(\phi)=0$,

(2) u countably additive

Note: u may not be monotone

Ex. u_1, u_2 measures on $\boldsymbol{a}$

Assume one of them is a finite measure.

Let $u(E)=u_1(E)-u_2(E)$ for $E\in\boldsymbol{a}$

Then u signed measure.

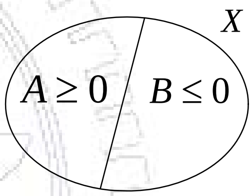
Major result: converse

i.e., every signed measure u is $u=u_1-u_2$, one of them is finite.

u signed measure on $\boldsymbol{a}$

Def: $E\in\boldsymbol{a}$ is positive ($E\ge 0$) if $u(A)\ge 0 \quad \forall A\in\boldsymbol{a},\ A\subseteq E$

$E\in\boldsymbol{a}$ is negative ($E\le 0$) if $u(A)\le 0 \quad \forall A\in\boldsymbol{a},\ A\subseteq E$



Thm 1. (Hahn decomposition of X)

u signed measure on X

Then $\exists A,B\in\boldsymbol{a} \ni A\ge 0,\ B\le 0,\ A\cup B=X$ & $A\cap B=\phi$

Lma 1.

(1) $E\ge 0,\ F\in\boldsymbol{a},\ F\subseteq E \Rightarrow F\ge 0$

(2) $E_n\ge 0\ \forall n \Rightarrow \bigcup_n E_n \ge 0$

(3) $E\ge 0,\ F\in\boldsymbol{a},\ F\subseteq E \Rightarrow u(F)\le u(E)$

Pf : (1) $\forall A\in\boldsymbol{a},\ A\subseteq F \Rightarrow A\subseteq E \Rightarrow u(A)\ge 0$

(2) $\forall A\in\boldsymbol{a},\ A\subseteq \bigcup_n E_n \Rightarrow A=(\bigcup_n E_n)\cap A=\bigcup_n (E_n\cap A)=\bigcup_n ((E_n\cap A)\setminus \bigcup_{i=1}^{n-1}(E_i\cap A))$

mutually disjoint

$$\Rightarrow u(A)=\sum_n u(\underbrace{(E_n\cap A)\setminus \bigcup_{i=1}^{n-1}(E_i\cap A)}_{\subseteq E_n})\ge 0.$$

$\therefore \bigcup_n E_n \ge 0$

(3) $\because u(E)=u(F)+u(E\setminus F)$ & $u(E\setminus F)\ge 0 \Rightarrow u(F)\le u(E)$.

Note: Similarly for negative sets: (3)$E\le 0, F\in\boldsymbol{a}, F\subseteq E\Rightarrow u(F)\ge u(E)$.

Lma 2. u signed measure

$E\subseteq F,\ E,F\in\boldsymbol{a},\ |u(F)|<\infty \Rightarrow |u(E)|<\infty$.

Pf : $\because u(F) = u(F \setminus E) + u(E)$.

If $u(E) = +\infty$, then $u : \boldsymbol{a} \to (-\infty, \infty]$

$\therefore u(F \setminus E)$ is in $(-\infty, \infty]$

$\Rightarrow u(F) = +\infty \qquad \to\leftarrow$

Similarly for $u(E) = -\infty$.

Pf of thm:

Assume, $u : \boldsymbol{a} \to (-\infty, \infty]$

Let $b = \inf\limits_{B_0 \leq 0} u(B_0)$

Then $\exists\, B_j \leq 0 \ni u(B_j) \to b$

Let $B = \bigcup\limits_j B_j \leq 0$, (by Lma 1 (2))

$\because b \leq u(B) \leq u(B_j) \to b$ (by (3))

$\Rightarrow 0 \geq b = u(B) > -\infty$

Let $A = X \setminus B$

Check: $A \geq 0$

Assume $A \ngeq 0$

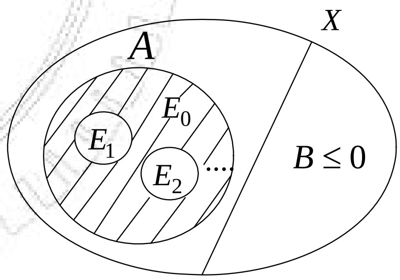
Then $\exists\, E_0 \subseteq A$, $E_0 \in \boldsymbol{a} \ni u(E_0) < 0$

Note: $E_0 \nleq 0$.

Reason: $E_0 \leq 0 \Rightarrow B \cup E_0 \leq 0$.

$\therefore b \leq u(B \cup E_0) = u(B) + u(E_0) = b + u(E_0) < b. \ \to\leftarrow$

$\Rightarrow \exists\, E_1 \subseteq E_0,\ E_1 \in \boldsymbol{a} \ni u(E_1) > 0$



(1) Let $m_1 \geq 1$ be the smallest $\ni u(E_1) \geq \dfrac{1}{m_1}$ & $E_1 \subseteq E_0$

$\because -\infty < u(E_0) < 0$

$\Rightarrow -\infty < u(E_1) < \infty$ (Lma. 2).

$\Rightarrow u(E_0 \setminus E_1) = u(E_0) - u(E_1) \leq u(E_0) - \dfrac{1}{m_1} < 0$

(2) $\therefore$ Let $m_2 \geq 1$ be the smallest $\ni \exists E_2 \subseteq E_0 \setminus E_1,\ u(E_2) \geq \dfrac{1}{m_2}$ (replacing E_0 by $E_0 \setminus E_1$)

$\vdots$

(k) Let $m_k \geq 1$ be the smallest $\ni \exists E_k \subseteq E_0 \setminus (\bigcup\limits_{i=1}^{k\text{-}1} E_i),\ u(E_k) \geq \dfrac{1}{m_k}$

Let $F_0 = E_0 \setminus (\bigcup\limits_k E_k)$. Check: $F_0 \leq 0$..(*)

Let $F \subseteq F_0$

$\because u(F) < \frac{1}{m_k - 1} \to 0$ as $k \to \infty$

(Reason: $F \subseteq E_0 \setminus (\bigcup_{i=1}^{k-1} E_i)\ \forall k$

$u(F) \geq \frac{1}{m_k - 1}$ for some k $\to\leftarrow$ minimality of m_k)

$\because \{E_k\}$ disjoint

$\Rightarrow u(\bigcup_k E_k) = \sum_k u(E_k)$

$\because \bigcup_k E_k \subseteq E_0$ & $-\infty < u(E_0) < 0 \Rightarrow \left| u(\bigcup_k E_k) \right| < \infty$ (by Lma 2)

$\therefore \sum_k u(E_k)$ converges

$\Rightarrow u(E_k) \to 0$ as $k \to \infty$

$\vee /$

$\frac{1}{m_k}$

$\Rightarrow \frac{1}{m_k} \to 0 \quad \Rightarrow m_k \to \infty$ as $k \to \infty$

$\Rightarrow u(F) \leq 0$

i.e., $F_0 \leq 0$

Note: $u(F_0) = u(E_0) - \sum_{k=1}^{\infty} u(E_k) \leq u(E_0) < 0$...............(**)

$\because B \cup F_0 \leq 0 \Rightarrow b \leq u(B \cup F_0) = u(B) + u(F_0) = b + u(F_0) < b \to\leftarrow$

$\Rightarrow A \geq 0$, completing the proof.

Note: Hahn decomposition not unique (cf: Ex.1.10.3).